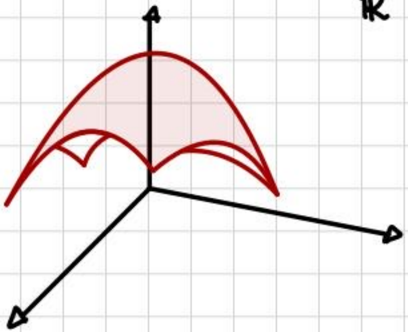
ANALYSIS II

1st YEAR BAI

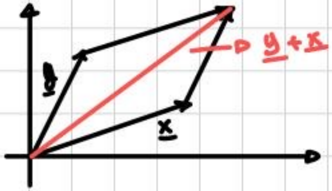
Written by:
Roberta Claps

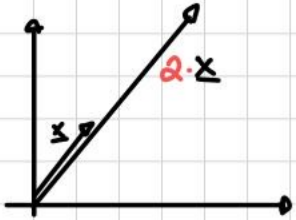
The Euclidean Space $\mathbb{R}^n$ (topology)

$z = f(x, y)$



$\mathbb{R}^n$ a prototype space





P_n is a vector Space

P_n = { set of all polynomials of degree n } which has a neutral element $\underline{0}$

$C^0(\mathbb{R})$ = {continous function in one variable with domain in $\mathbb{R}$}

Dot product : $x \cdot y = \sum_{i=1}^{d} x_i y_i$ the dot product of two vectors is a scalar

$u \perp v \Leftrightarrow u \cdot v = 0$

$\|v\| = \sqrt{\Sigma v_i} \longrightarrow$ Chauchy - Schwarz Inequality

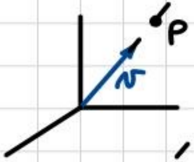
Straight line in $\mathbb{R}^3$ $\quad \underline{v} = (\alpha, \beta, \gamma)$

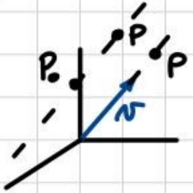
$$P \in \mathbb{R}^3 = (x, y, z) = t \cdot \underline{v}$$

$\downarrow$

T-expansion: any Real number

$$\begin{cases} x = t \cdot \alpha \\ y = t \cdot \beta \\ z = t \cdot \gamma \end{cases}$$



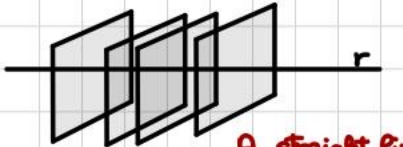


$$P - P_0 = t \cdot \underline{v}$$

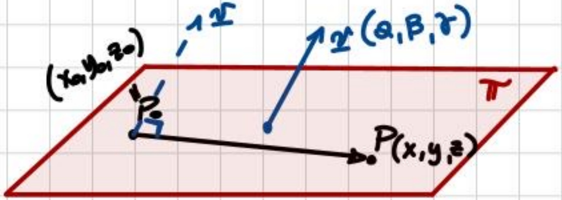
$$\begin{cases} x = x_0 + \alpha t \\ y = y_0 + \beta t \\ z = \gamma_0 + \gamma t \end{cases}$$

Planes in $\mathbb{R}^3$

Plane crossing $P_0(x_0, y_0, z_0)$ with direction $\underline{v}(\alpha, \beta, \gamma)$



A straight line identifies a family of planes



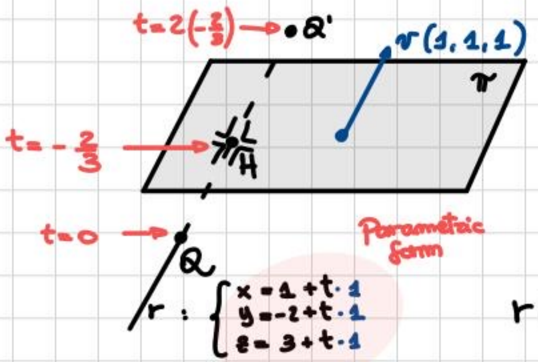
$$\underline{P} \in \pi \iff (\underline{P} - \underline{P_0}) \perp \underline{v} \iff (\underline{P} - \underline{P_0}) \cdot \underline{v} = 0$$

$$\alpha(x-x_0)+\beta(y-y_0)+\gamma(z-z_0)=0$$

$$\underbrace{\alpha x+\beta y+\gamma z}_{\text{linear combination}}=\alpha x_0+\beta y_0+\gamma z_0$$

Exercise: π $\mathbb{R}^3$ and $Q \notin \pi$

Determine the symmetric Q' of Q with respect to π.



$$r:\begin{cases} x=1+t\cdot 1\\ y=-2+t\cdot 1\\ z=3+t\cdot 1\end{cases}$$

$Q\,(1,-2,3)$

$\pi\;\; x+y+z=0$

$$r\cap\pi=\begin{cases} x=1+t\\ y=-2+t\\ z=3+t\\ x+y+z=0\end{cases} \Rightarrow 1+t-2+t+3+t=0;\; 2+3t=0,\; t=-\frac{2}{3}$$

↓ H is reached at time:

$\Rightarrow H\left(1-\frac{2}{3};\, -2-\frac{2}{3};\, 3-\frac{2}{3}\right)$ but we need Q'

$\Rightarrow Q'\left(1-\frac{4}{3};\, -2-\frac{4}{3};\, 3-\frac{4}{3}\right)$

$$\begin{cases} x_1 - 3x_2 - 6x_3 = 1 \\ x_1 + x_2 - 4x_3 = 5 \\ 2x_1 - 10x_2 - 11x_3 = 0 \end{cases} \Rightarrow \text{Impossible} \Rightarrow \text{these 3 planes has no points in common}$$

Exercise

$$\left(\begin{array}{ccc|c} 1 & -3 & -5 & 0 \\ 1 & -2 & -4 & 3 \\ 2 & -7 & -11 & -3 \end{array}\right)$$

$$\begin{array}{c} \\ II - I \\ III - 2I \end{array} \left(\begin{array}{ccc|c} 1 & -3 & -5 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & +3 \end{array}\right) \quad 3II + I \left(\begin{array}{ccc|c} 1 & 0 & -2 & 9 \\ 0 & 1 & 1 & 3 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right)$$

x_1 has the pivot

x_2 has the pivot

x_3 has not

the solution is a line that depends on x_3 since it has No pivot : $\begin{cases} x_1 - 2x_3 = 9 \\ x_2 + x_3 = 3 \end{cases}$ $t = x_3$

static representation of a straight line as an intersection between two planes

$$\Rightarrow \begin{cases} x_1 = 9 + 2t \\ x_2 = 3 - t \\ x_3 = t \end{cases}$$

Exercise

$$\begin{cases} -x - y = 0 \\ x - 3y - z = 0 \\ x + \frac{1}{2}y = 0 \\ 3x - 2y + \frac{1}{2}z = 0 \end{cases}$$

Homogenous

$$\begin{cases} -x_1 + x_2 - 2x_3 = 1 \\ x_3 - x_4 = 0 \end{cases}$$

$$\left(\begin{array}{cccc|c} -1 & 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right)$$

$\downarrow$ x_2 and x_4 has no pivot

$x_2 = t$

$x_4 = \tau$

$$\begin{cases} x_1 = 1 \cdot t - 2\tau - 1 \\ x_2 = t \\ x_3 = \tau \\ x_4 = \tau \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \tau \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix} \in \mathbb{R}^4$$

The Euclidean Space

Dot product: $x, y \in \mathbb{R}^d$

$$x \cdot y = \sum_{i=1}^{d} x_i \cdot y_i \qquad x \cdot y \in \mathbb{R}$$

properties:

- symmetry: $x \cdot y = y \cdot x$
- linearity with respect to the first variable: $(ax + by) \cdot z = a(x \cdot z) + b(y \cdot z)$
- positive definite: $x \cdot x \geq 0$ and $x \cdot x = 0$ if and only if $x = 0$

Proof

$$\sum (\alpha x_i + \beta y_i) \cdot z_i = \sum \alpha x_i z_i + \sum \beta y_i z_i = \alpha (x \cdot z) + \beta (y \cdot z) \quad \square$$

Proof: $x \cdot x = 0 \Longleftrightarrow x = 0$

$\Longleftarrow$) Trivial. $x = 0 \Rightarrow \sum x_i \cdot x_i = 0 + 0 + \dots + 0 = 0$

$\Longrightarrow$) $0 = \sum_{i=1}^{d} x_i^2$ the only way to obtain 0 from positive sum is that $x_i = 0 \ \forall i = 1, \dots, d$ $\square$

The relation with norm: $\|x\| = \sqrt{x \cdot x}$

$\|x\| = \sqrt{\sum_{i=1}^{d} x_i^2}$ the norm provides the length of the arrow representing the vector

$\Rightarrow d(x, y) = \|x - y\|$ every abstract space provided by an inner product can provide a norm that can provide a distance.

Chauchy-Schwarz Inequality

$$\boxed{|x \cdot y| \leq \|x\|\|y\|}$$

~~And the equality holds if and only if x and y are linearly dependent~~.

Proof: $\|x+y\|^2 = (x+y)\cdot(x+y) \geq 0$

$\big\downarrow$ linearity

$$= \underline{x}\cdot(\underline{x}+\underline{y}) + \underline{y}\cdot(\underline{x}+\underline{y}) = \underline{x}\cdot\underline{x} + \underline{x}\cdot\underline{y} + \underline{y}\cdot\underline{x} + \underline{y}\cdot\underline{y} = \|x\|^2 + 2(\underline{x}\cdot\underline{y}) + \|y\|^2$$

$$\|x+ty\|^2 = \|\underline{x}\|^2 + 2t(\underline{x}\cdot\underline{y}) + t^2\|y\|^2 = p(t)$$

hence, the sign of the discriminant is given $4(x\cdot y)^2 - 4\|x\|^2\|y\|^2 \leq 0$

$$(x\cdot y)^2 \leq \|x\|^2\|y\|^2$$

$$\Rightarrow |x\cdot y| \leq \|x\|\|y\| \qquad \square$$

Now proving the equality:

$|x\cdot y| = \|x\|\|y\| \Leftrightarrow$ x and y are linearly dependent. $\underline{x} = k\cdot\underline{y}$

$\Rightarrow \Delta$ of $p(t) = 0 \quad \therefore \exists\, t_0$ s.t. $p(t_0) = 0$

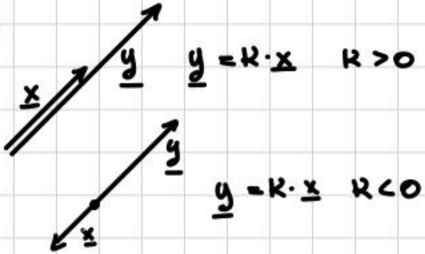
$p(t_0) = \|x + t_0 y\|^2 = 0 \Rightarrow \underline{x} = t_0\,\underline{y}$ so $\underline{x}$ and $\underline{y}$ are colinear.

$\Leftarrow)$ $|Ky \cdot y| = \|Ky\| \cdot \|y\|$ Direct Proof

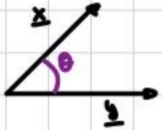
$$\underbrace{|K|\,|y \cdot y|}_{|K|\,\|y\|^2} = |K|\,\|y\|^2$$

$$|K|\,\|y\|^2 = |K|\,\|y\|^2 \quad \square$$

Graphically



Consequence: $-1 \leq \dfrac{x \cdot y}{\|x\| \cdot \|y\|} \leq 1$ allowed to define angles between vectors



$\theta = \arccos\left(\dfrac{x \cdot y}{\|\underline{x}\|\,\|\underline{y}\|}\right)$

the sign of each angle is not explicit

Triangular Inequality

$$\|x+y\| \le \|x\| + \|y\|$$

$$\|\underline{x}+\underline{y}\|^2 = (\underline{x}+\underline{y})\cdot(\underline{x}+\underline{y}) \underset{\text{by linearity}}{=} \underline{x}\cdot\underline{x} + 2(\underline{x}\cdot\underline{y}) + \underline{y}\cdot\underline{y} = \underbrace{\|x\|^2 + 2(x\cdot y) + \|\underline{y}\|^2}_{(\|\underline{x}\|+\|\underline{y}\|)^2}$$

Fundamental Property of the norm

Let $x, y \in R^d$ and $\alpha \in R$. Then

- positivity $\|\underline{x}\| > 0$ and $\|x\| = 0 \Leftrightarrow x = 0$ (the proof pos. is given by pos. of dot product)
- homogeneity $\|\alpha\underline{x}\| = |\alpha|\|\underline{x}\|$ (pseudo-homogeneity)
- triangle inequality $|\underline{x}+\underline{y}| \le \|\underline{x}\| + \|\underline{y}\|$

homogeneity proof: $\|\alpha x\| = \sqrt{(\alpha x)\cdot(\alpha x)} = \sqrt{\alpha^2 x\cdot x} = |\alpha|\sqrt{x\cdot x} = |\alpha|\,\|\underline{x}\|$

homogeneity $\Rightarrow$ "re-scaling" is possible

Pythagorean Theorem

Consider two orthogonal vectors $x, y \in \mathbb{R}^d$ (i.e. $x \cdot y = 0$). then,

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2$$

$$\|\underline{x}+\underline{y}\|^2 = (\underline{x}+\underline{y})\cdot(\underline{x}+\underline{y}) = \underline{x}\cdot\underline{x} + \underbrace{2(x\cdot y)}_{0} + \underline{y}\cdot\underline{y} \Rightarrow \|\underline{x}\|^2 + \|\underline{y}\|^2 \qquad \square$$

① $v_1(2,0,1)$, $v_2(-2,1,0)$, $v_3(4,-3,2)$ edges and angles

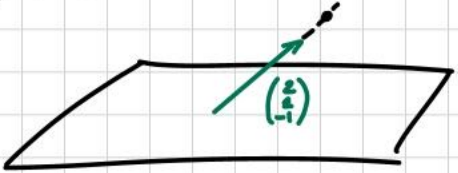
$\overline{v_1 v_2} = \|v_1 - v_2\| = \|(0,-1,1)\| = 2$

$\overline{v_2 v_3} = \|v_3 - v_2\| = \|(2,-4,2)\| = 24$

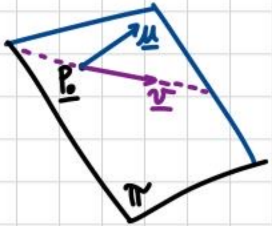
$\overline{v_1 v_3} = \|v_3 - v_1\| = \|(2,-3,1)\| = 14$

② straight line in $\mathbb{R}$ passing by $(2,1,-3)$;

plane : $2x + y - z = 5$



What if $\mathbb{R}^4$?



$\underline{u}$, $\underline{v}$ not necessarely orthogonal

A plane in $\mathbb{R}^4$ has 2 orthogonal vectors

$$\begin{bmatrix}x\\y\\z\\w\end{bmatrix} = \begin{bmatrix}x_0\\y_0\\z_0\\w_0\end{bmatrix}_{P_0} + t\underset{\underline{v}}{\begin{bmatrix}\alpha\\\beta\\\gamma\\\delta\end{bmatrix}} + \tau\underset{\underline{u}}{\begin{bmatrix}\alpha^*\\\beta^*\\\gamma^*\\\delta^*\end{bmatrix}}$$

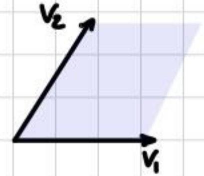
cartesian expression of a plane : $\begin{cases} ax + by + cz + dw = e \\ a^*x + b^*y + c^*z + d^*w = e^* \end{cases}$

③ $\begin{cases} x + y + z = 0 \\ 2x + y + z = 0 \end{cases}$

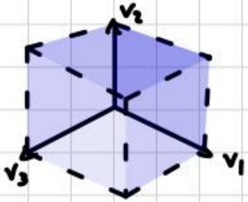
$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right)$ $\quad \mathrm{II} - 2\mathrm{I} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & +1 & +1 & 0 \end{array}\right) \Rightarrow z$ is the parameter : $\begin{cases} x = 0 \\ y = -t \\ z = t \end{cases}$

Cross Product $\mathbb{R}^3$

recursive def. of det A with $A \in \mathcal{M}(m \times n)$ $\det A = \sum_{j=1}^{n} a_{ij}(-1)^{(i+j)} \cdot \det(A_{ij})$



the det. of a 2x2 matrix is the area between the two vectors columns



In 3x3 it represents the volume

det A = 0 if two columns are Linearly Dependent

Cross Product Definition: in $\mathbb{R}^3$

$$x \times y = \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_2 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}$$

we can express the different components as determinants: $(x \times y)_1 = \det\begin{pmatrix} x_2 & y_2 \\ x_3 & y_3 \end{pmatrix}$, $(x \times y)_2 = -\det\begin{pmatrix} x_1 & y_1 \\ x_3 & y_3 \end{pmatrix}$ and $(x \times y)_3 = \det\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$

$$\underline{x} \times \underline{y} = \det\begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} = \underline{i} \cdot \underbrace{\det\begin{pmatrix} x_2 & x_3 \\ y_2 & y_3 \end{pmatrix}}_{(x \times y)_1} - j \cdot \underbrace{\det\begin{pmatrix} x_1 & x_3 \\ y_1 & y_3 \end{pmatrix}}_{(x \times y)_2} + k \cdot \underbrace{\det\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}}_{(x \times y)_3}$$

$u\,(1, 2, 3)$
$v\,(-1, 1, 4)$

$$\underline{u} \times \underline{v} = \det\begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 3 \\ -1 & 1 & 4 \end{pmatrix} = \underbrace{\underline{i}\cdot 5 - \underline{j}\cdot 7 + \underline{k}\cdot 3}_{\underline{u}\times\underline{v}}$$

Dot product to see wether they are orthogonal or not:

$$\left.\begin{array}{c} \underline{u} \bullet (\underline{u}\times\underline{v}) \\ \Vert \\ 1\cdot 5 + 2(-7) + 3\cdot 3 = 0 \\ \\ \underline{v} \bullet (\underline{u}\times\underline{v}) \\ \Vert \\ -1\cdot 5 - 7 + 4\cdot 3 = 0 \end{array}\right\} \begin{array}{l} \underline{u} \perp (\underline{u}\times\underline{v}) \\ \underline{v} \perp (\underline{u}\times\underline{v}) \end{array}$$

The output of the cross product is always orthogonal to the vectors generators.

example: $\underline{e_1}, \underline{e_2}$

$$\underline{e_1} \times \underline{e_2} = \det\begin{pmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \underline{i}\cdot 0 - \underline{j}\cdot 0 + \underline{k}\cdot 1 = e_3$$

Characterisation of the Cross-Product

If $x, y, z \in \mathbb{R}^3$, then

$$\det(x, y, z) = (x \times y) \bullet z$$

Proof:

$$(x \times y) \cdot z = \det\begin{pmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \bullet z = \underbrace{z_1 \cdot \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + z_2 (-1) \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + z_3 \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}}$$

$$= \det\begin{pmatrix} z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$$

Geometric properties of the cross product

- $x \times y$ is orthogonal to both x and y
- $\|x \times y\| = \|x\| \|y\| |\sin\theta|$
- $\det(x, y, x \times y) \geq 0$, if $\det(x, y, x \times y) \neq 0$ then $(x, y, x \times y)$ is a basis with positive orientation

Proof:

①

② $\underset{z}{\underset{\parallel}{\underline{x}}} \times \underline{y} \Rightarrow \quad V = \underbrace{\left| \det\begin{pmatrix} z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \right|}$ = area of the basis $\cdot \underset{\|z\|}{\underset{\parallel}{h}}$ $\Rightarrow \|z\| \|x + y\| =$ area of the basis $\cdot \|z\|$

$\Rightarrow \|x + y\| =$ area of the basis

they are colinear

this is the equality case of the Schwarz-Inequality $z \bullet (x \times y) = \|z\| \|x \times y\|$

Subsets in $\mathbb{R}^d$, portions $\mathbb{R}^d$

- '<u>Dynamically</u>': result of parametrization

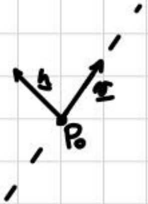
$$D = \left\{(x,y,z)\in\mathbb{R}^3, \text{ s.t. } \exists\, t\in\mathbb{R} \ \ (x(t), y(t), z(t))\right\}$$

output through a function

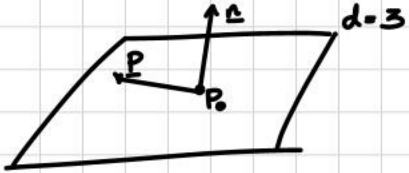
- '<u>Statically</u>': $D = \left\{(x,y,z)\in\mathbb{R}^3 \text{ s.t. } x+2y-3z \overset{\le}{\overset{\text{or}}{=}} 1\right\}$

Shapes:

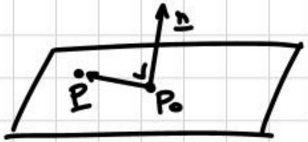
- Straight lines: $\underline{P} = \underset{\text{point}}{P_0} + \overset{\text{expansion}}{t}\,\underset{\text{direction}}{\underline{v}}$



d = 2



- Planes:
  if d > 3



in d = 3 we describe an object with 2 degree of freedom
with d > 3 we have d-1 deg. of freedom

$V = \mathbb{R}^3 \quad W = \{(x,y,z) \in V \text{ s.t. } x + y - 2z = 0\}$ basis $S \left\{ \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \end{pmatrix} \right\} \to$ 2 basis

$V = \mathbb{R}^4 \quad W = \{(x,y,z,q) \in V \text{ s.t. } x + y - 2z - q = 0\}$ basis $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \to$ 3 basis

- Circles: $\underline{P}_0 = (x_0, y_0)$

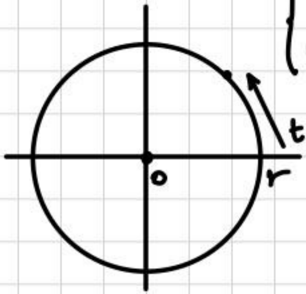
$\underline{P} = (x, y)$

$\text{dist}(P, P_0) = \|\underline{P} - \underline{P}_0\| = \sqrt{(x - x_0)^2 + (y - y_0)^2} = r$

$D = \{(x,y) \in \mathbb{R}^2 \text{ s.t. } (x - x_0)^2 + (y - y_0)^2 = r^2$

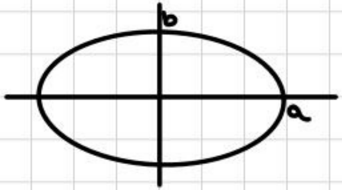
$D = \{(x_0 + r\cos t,\ y_0 + r\sin t) : t \in \mathbb{R}\}$

$$\begin{cases} x = \overset{r}{\uparrow}\cos t + x_0 \\ y = \overset{r}{\uparrow}\sin t + y_0 \end{cases}$$



- Ellipses:

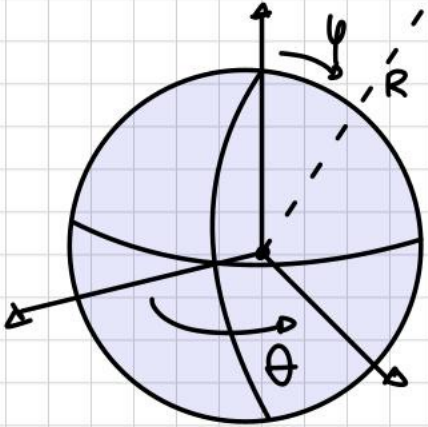
$$\begin{cases} x = a\cos t \\ y = b\cos t \end{cases} \longrightarrow \text{reshaping a circumference}$$



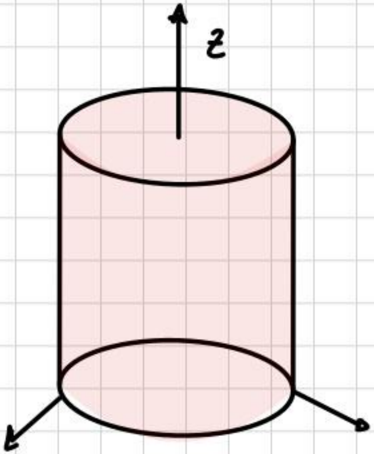
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

- Sphere:

$$\{(x,y,z) \in \mathbb{R}^3 : (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2\}$$



- Cylinder



$$x^2 + y^2 = r^2$$

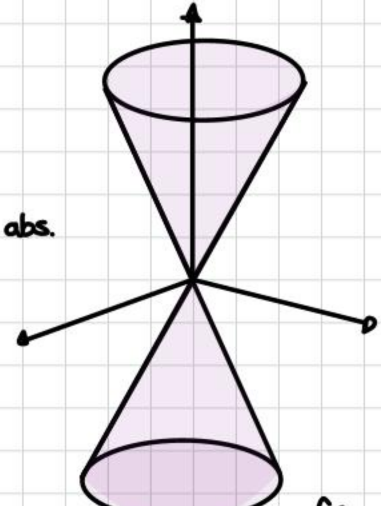
- Cones

$y = |x| = \sqrt{x^2}$

$\Rightarrow z = \sqrt{x^2 + y^2}$ the root of the square is the abs.

just upper part of the cone

Thus: $|z| = \sqrt{x^2 + y^2}$



Natural Domain

$$z = \frac{\sqrt{(x^2-2x-y)(x^2-2x+y)^2}}{\left(x-\frac{3}{2}\right)^2+\left(y-\frac{1}{2}\right)^2} + \ln\left(\frac{x+1}{2-x}\right)$$

$$\begin{cases} \left(x-\frac{3}{2}\right)^2+\left(y-\frac{1}{2}\right)^2 \neq 0 \\ (x^2-2x-y)(x^2-2x+y)^2 \geq 0 \\ x \neq 2 \\ \frac{x+1}{2-x} > 0 \end{cases}$$

$x \neq \frac{3}{2}$ $y \neq \frac{1}{2}$

$\Rightarrow$ $x^2-2x \geq y$ parabola

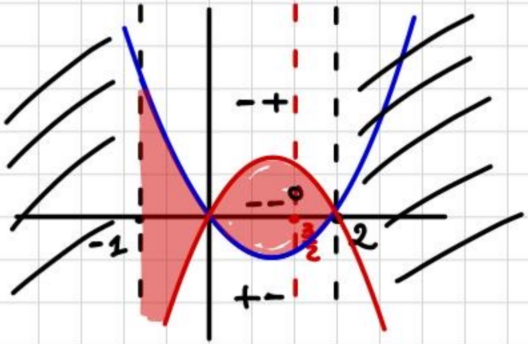
$x^2-2x \geq -y$

$y \geq -x^2+2x$

$\frac{x+1}{2-x} > 0 \rightarrow x > -1$, $x < 2$

$$f(x,y) = \frac{\sqrt{(x^2-2x-y)(x^2-2x+y)^2}}{\left(x-\frac{3}{2}\right)^2+\left(y-\frac{1}{2}\right)^2} + \ln\left(\frac{x+1}{2-x}\right)$$

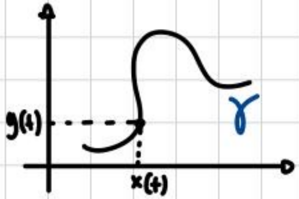
Domain:



Parametric Curves in $\mathbb{R}^d$

(x, y) denotes the position of a point in 2-d _
If the point is moving, the position will depend on time

$$\underline{P} = \begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

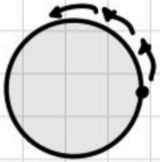


example: A straight line → position changes linearly wrt time

$$\begin{cases} x(t) = a + mt \\ y(t) = b + mt \end{cases}$$

p_0 v_0

example: Circle



$$\begin{cases} x = \cos t \\ y = \sin t \end{cases} \qquad t \in [0, 2\pi]$$

The derivative of the position $r'(t)$ is the velocity of the point

(the graph γ lives in $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2$
the position lives in the codomain → the curve is the Im. set of $f(x)$)

DEF PARAMETRIC CURVE: every point $x \in \mathbb{R}^d$ of the codomain if $\exists\ t$ s.t. $x = \gamma(t)$

- Continuity and differentiability of γ: A curve is continuous at a point $\underline{P_0} = \gamma(t_0)$ if all the coordinate functions continuous at t_0.

DA CONTINUARE

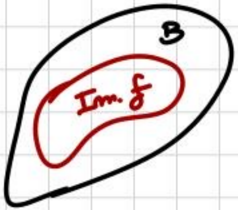
Going back to basis:

function: a map from a set to another

$$A \longrightarrow B$$

$$\underline{x} \longrightarrow y = f(x) \quad \text{s.t. } y!$$

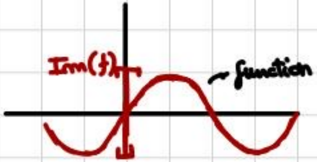
$$Im(f) = \{ b \in B \text{ s.t. } \exists \; x \in A , f(x) = b \}$$



A graph (Instead) is a portion of A×B

example: $y = \sin x : R \to R$

$$im(f) = [-1,1] \subseteq R$$



$f : R^n \to R^m$ linear

$$\underline{x} \longrightarrow \underline{w} = f(\underline{x}) = A \cdot \underline{x} \quad \to \quad Im(f) \underset{SSP}{\subseteq} R^m \quad \to \quad 0 \le \overbrace{\dim(Im(f))}^{rank(A)} \le m$$

Determine the tangent to the image of $t \to (t^2, t^3)$

$t \longmapsto \gamma(t) = (t^2, t^3)$ each components $\gamma_1, \gamma_2 \in C^\infty(\mathbb{R})$ every polynomial can be derived infinite times

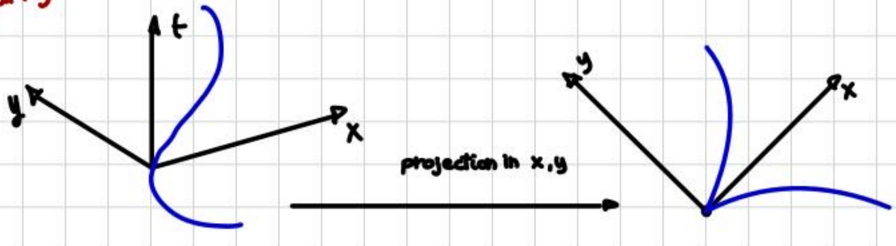
$\gamma'(t) = (2t, 3t^2)$ $\boxed{\gamma'(0) = \underline{0}}$ Alert for lack of regularity

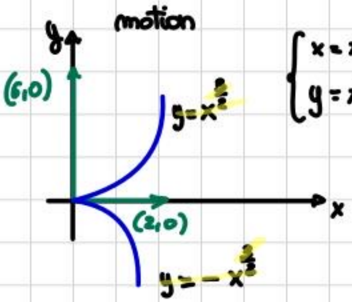
$\gamma''(t) = (2, 6t)$ $\gamma''(0) = (2,0)$

$\gamma'''(t) = (0, 6)$ $\gamma'''(0) = (0,6)$

Not colinear

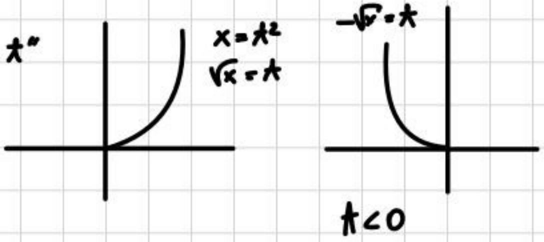
it vanishes at a point





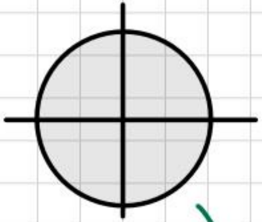
$$\begin{cases} x = t^2 \\ y = t^3 \end{cases}$$

"eliminate t"



$\Rightarrow t = \pm\sqrt{x}$

$\Rightarrow y = t^3 = (\pm\sqrt{x})^3 = \pm\, x^{\frac{3}{2}}$



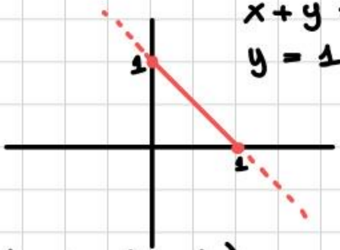
$$\begin{cases} x = \cos t \\ y = \cos t \end{cases}$$

$$\begin{cases} x = \cos^2 t \\ y = \sin^2 t \end{cases}$$

$\downarrow$

$\gamma'(-2\cos t \sin t\ ,\ 2 \sin t \cos t)$

$\gamma'(0) = 0$



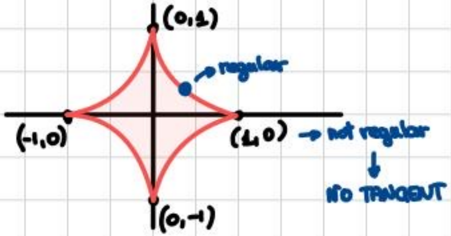
$x + y = 1$

$y = 1 - x$

pattern for odd powers

pattern for even powers

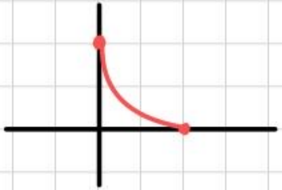
$$\begin{cases} x = \cos^3 t \\ y = \sin^3 t \end{cases}$$



$\gamma' = (-3\cos^2 t \sin t\ ,\ 3\sin^2 t \cos t)$

NO REGULAR = NO TANGENT

$$\begin{cases} x = \cos^4 t \\ y = \sin^4 t \end{cases}$$



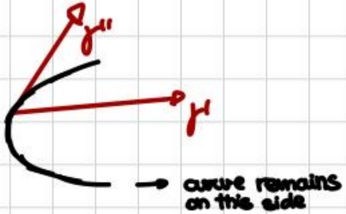
Inflection point IS regular

Definition: Regular Points

If $\gamma'(t_0) \neq 0$ then γ admits Taylor Expansion

$$\gamma(t_0+h) = \gamma(t_0) + h\gamma'(t_0) + o(h)$$

thus, $\operatorname{Im}(\gamma)$ is close to the line $h \mapsto \gamma(t_0) + h\gamma'(t_0)$: this is the tangent line
This is called regular point



Biregular: If $\gamma'(t_0)\neq 0$ and $\underline{\gamma''(t_0) \text{ not colinear to } \gamma'(t_0)}$ then

$$\gamma(t_0+h) = \gamma(t_0) + h\gamma'(t_0) + \boxed{\frac{h^2}{2}}\gamma''(t_0) + o(h)$$

thus, in the frame $(\gamma'(t_0), \gamma''(t_0))$ centered at $\gamma(t_0)$, $\operatorname{Im}(\gamma)$ is close to the parab. $y = \boxed{\frac{x^2}{2}}$

Inflection Points: the case where $\gamma'(t_0)$ colinear to $\gamma''(t_0)$

$$\gamma(t_0+h) = \gamma(t_0) + \left(h + \frac{h^2}{2}\lambda\right)\gamma' + \boxed{\frac{h^3}{6}}\gamma'''(t_0) + \sigma(h^3)$$

$$\|$$

$$\underbrace{\gamma'(t_0)h + \frac{\gamma''(t_0)\,h^2}{2}}_{\text{colinear}} = \gamma'(t_0)\left(h + \frac{h^2}{2}\lambda\right)$$

$\Rightarrow$ because $\gamma''(t_0)$ can be writen as an expansion of $\gamma'(t_0)$

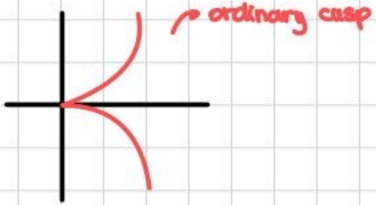
the $\operatorname{Im}(\gamma)$ is close to the curve $\frac{x^3}{6}$

Ordinary Cusp If $\gamma(t_0)=0$ but $\gamma''(t_0)$ and $\gamma'''(t_0)$ not colinear

t_0+h-t_0

$$\gamma(t_0+h)=\gamma(t_0)+\frac{h^2}{2}\gamma''(t_0),+\frac{h^3}{6}\gamma'''(t_0)+o(h^3)$$

The image is closed to
$h \longmapsto (h^2, h^3)$



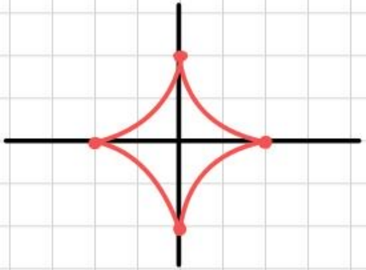
$$\begin{cases} x=\cos^3 t \\ y=\cos^3 t \end{cases}$$

draw its Image and study its cusp

$\gamma'=(-3\cos^2 t\sin t,\ 3\sin^2 t\cos t)$

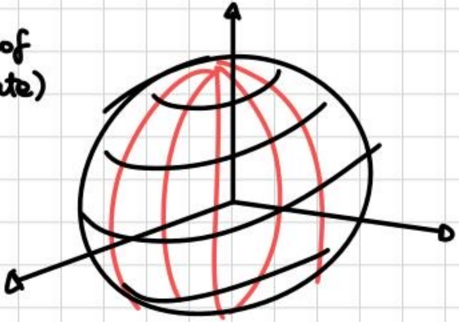
$\gamma''=(+6\cos t\sin t-3\cos^3 t,\ 6\sin t\cos^2 t-3\sin^3 t)$ → $\gamma''(0)=(-3,0)$

$\gamma'''=(-6\sin^2 t+6\cos^2 t+9\cos^2 t\sin t,\ 6\cos^3 t-12\cos t\sin t-9\sin^2 t\cos t)$ → $\gamma''(0)=(0,6)$



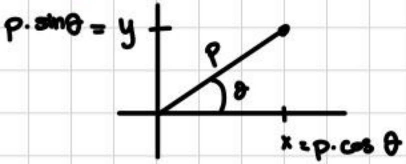
$\gamma''(0)$ and $\gamma'''(0)$ not colinear

Parametric Form of (polar sphere coordinate)

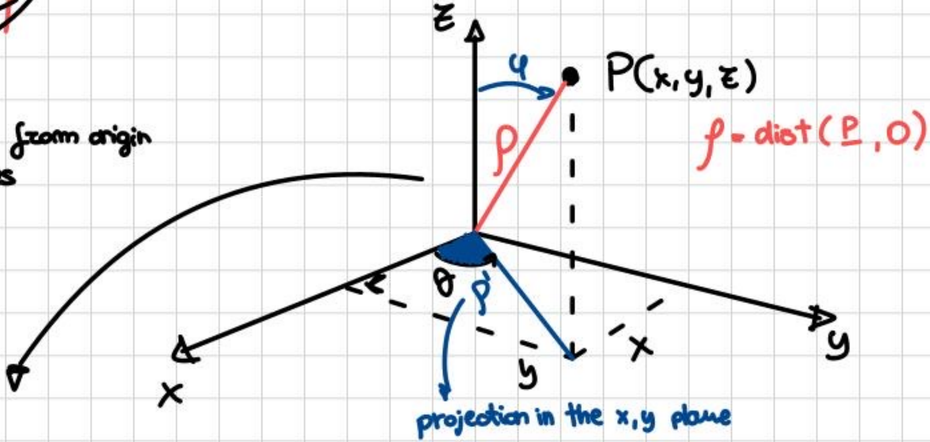


Polar Coordinates in $\mathbb{R}^2$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$



I want to identify: position → distance from origin
+ 2 angles



I want to write x,y,z with ρ, θ, φ

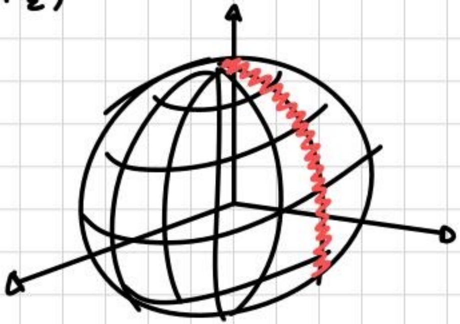
$$\Rightarrow z = \rho\cos\varphi$$
$$x = \rho\sin\varphi\cos\theta$$
$$y = \rho\sin\varphi\sin\theta$$

$$\theta \in [0, 2\pi]$$
$$\varphi \in [0, \pi]$$
$$\rho \in [0, +\infty)$$

$$\overset{\text{domain}}{A} = [0,2\pi] \times [0,\pi] \times [0,+\infty)$$

going back to the sphere: $\gamma(\theta, \varphi)$

Fixing $\underline{\varphi}$: $\gamma(\theta, \frac{\pi}{2})$



Fixing $\underline{\theta}$: $\gamma(\frac{3}{2}\pi, \varphi)$

A note on the Lagrange value theorem →

- f continous in $[a,b]$ differentiable in (a,b) $\Rightarrow \dfrac{f(b)-f(a)}{b-a} = f'(c)$
- f continous in $[a,b]$ differentiable in $[a,b]$ $\Rightarrow f(b)-f(a) = \int_a^b f(t)\,dt$

Formula including integral are always weaker than formula involving derivatives

- $\gamma : \mathbb{R} \to \mathbb{R}^2$ defined by

$$\gamma(t) = (\cos(t), \sin(t))$$

$t \in [0, 2\pi]$ $\quad \gamma(0) = (1,0) = \gamma(2\pi)$

$$\frac{\gamma(2\pi)-\gamma(0)}{2\pi} = \underline{0} \qquad \exists\, t_0 \text{ s.t. } \gamma'(t_0) = \underline{0}\ ?$$

NO → the speed is 1 costantly, never 0

Look at what comes component-wise.

$\gamma_1 = \cos t \in C^\infty([0,2\pi])$ $\quad \gamma_1(0) = \gamma(2\pi)$ $\quad \exists\, t_1 \in [0,2\pi]$ s.t. $\gamma_1'(t_1) = 0$? YES

$\gamma_2 = \sin t$ $\quad \exists\, t \in [0, 2\pi]$

Study the cycloid curve $\begin{cases} x(t) = t - \sin t \\ y(t) = 1 - \cos t \end{cases}$ $t \geq 0$

describing the motion of a point on a circumference that rolls without crawling

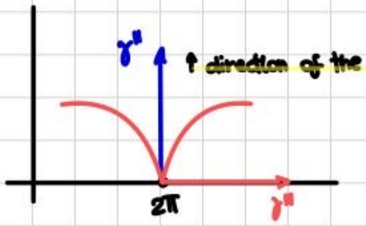
$\gamma' = (1 - \cos t, + \sin t) = 0 \rightarrow t = k \cdot 2\pi$

$\gamma'' = (+\sin t, +\cos t) \rightarrow \gamma''(2\pi) = (0, 1)$

$\gamma''' = (\cos t, -\sin t) \rightarrow \gamma'''(2\pi) = (1, 0)$

$\gamma'(t) = 0$

Ex $t = 2\pi$ $\gamma(2\pi) = (2\pi, 0)$



↑ direction of the cusp

there is a cusp for every solution at $\gamma'(t) = 0$

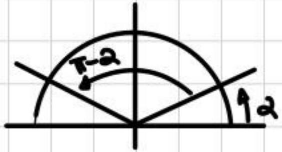
Exercise 8

Let $\gamma : [0,\pi] \to \mathbb{R}^2$ be the parametric curve defined by

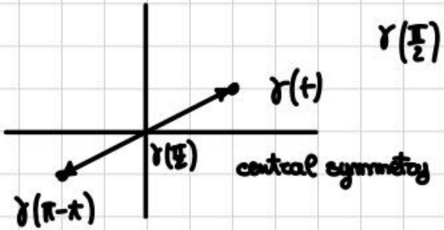
$$\gamma(t) = (\cos(3t), \sin(2t))$$

a) Using a transformation trying to obtain $\mathrm{Im}(\gamma)$ on $[\frac{\pi}{2}, \pi]$ from $[0, \pi/2]$:

$$\begin{aligned}(\cos 3(\pi-t), \sin 2(\pi-t)) &= (\cos(3\pi-3t), \sin(2\pi-2t))\\ &= (\cos(\pi-3t), \sin(-2t))\\ &= (-\cos(3t), -\sin(2t))\\ &= \gamma(\pi-t)\end{aligned}$$



$\Rightarrow \cos(\pi-3t) = -\cos(3t)$

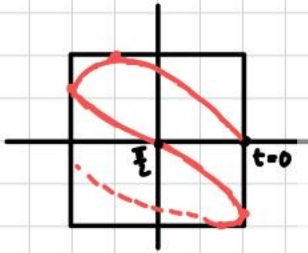


$\gamma(\frac{\pi}{2}) - \gamma(-\frac{\pi}{2}) = 0$

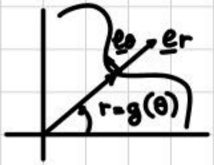
b) Find the values of $t \in [0, \pi/2]$ for which γ' is parallel to the horizontal or vertical axes

$\gamma' = \begin{pmatrix}-3\sin 3t\\ 2\cos 2t\end{pmatrix}$ if γ' is like $\begin{pmatrix}0\\ \mathbb{R}\end{pmatrix}_{>0}$ $\uparrow$ $\sin 3t = 0$ $\quad 3t = n\pi \quad t = \frac{n}{3}\pi$

if γ' is like $\begin{pmatrix}\mathbb{R}\\ 0\end{pmatrix}^{>0} \longrightarrow$ $\cos 2t = 0 \quad t = +\frac{\pi}{4} + n\frac{\pi}{2}$ ...



Polar Coordinates



$x = r\cos\theta$
$y = r\sin\theta$ e_r

$\|\underline{e}_r\| = 1$

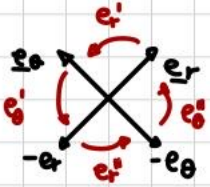
$x = -\sin\theta$
$y = \cos\theta$ e_θ $\|e_\theta\| = 1$

$$\underline{e}_r \bullet \underline{e}_\theta = -\sin\theta\cos\theta + \sin\theta\cos\theta = 0$$

$\underline{e}_\theta$ and $\underline{e}_r$ are orthonormal basis

$$\gamma(\theta) = \begin{pmatrix} g(\theta)\cos\theta \\ g(\theta)\sin\theta \end{pmatrix} \Rightarrow \gamma(\theta) = g(\theta)\, e_r$$

⇒ Properties: - $e_r \bullet e_\theta = 0 \quad \forall\, \theta \geq 0$
- $e_r' = e_\theta$ and $e_\theta' = -e_r$



the derivative rotates the functions e_r, e_θ

$$\underline{e}_r = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad \underline{e}_r = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \underline{e}_\theta \qquad e_\theta' = \begin{pmatrix} -\cos\theta \\ -\sin\theta \end{pmatrix} = -\underline{e}_r$$

THM: $g : I \to [0,+\infty)$ $\quad g \in C^2$ we define $\gamma : I \to \mathbb{R}^2$ $\qquad \gamma = g e_r$

Then γ is of class C^2 and $\gamma' = g'e_r + g e_\theta$ and $\gamma'' = (g''-g)\, e_r + 2g' e_\theta$

Exercise 9

Find the image of the following curves, assigned in polar form $r = g(\theta)$

a) $r = 1$ circle

b)

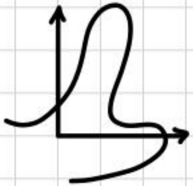
$$r = 1 - \sin(2\theta)$$

$$\gamma' = \underbrace{g'\underline{e}_r + g\,\underline{e}_\theta}_{\text{independent}} = 0$$

$\Rightarrow$ It is equal 0 iff $g = g' = 0$

Topology

Limits:
discretise time



A sequence in $\mathbb{R}^d$ as any
$f : \mathbb{N} \to \mathbb{R}^d$
$\downarrow$
$\underline{p} = \underline{X} \in \mathbb{R}^d$
↳ the position in $\mathbb{R}^d$ which depends on n

collection of points depending on n
$\uparrow$
$(\underline{X}_n)_{n \in \mathbb{N}}$

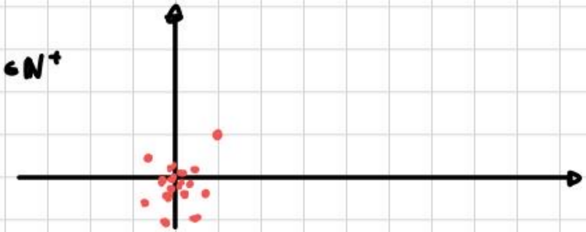
Representing:

1) $\left(\frac{1}{n}\cos n, \frac{1}{n}\sin n\right)_{n \in \mathbb{N}^+}$

$x = \frac{1}{n}\cos n$

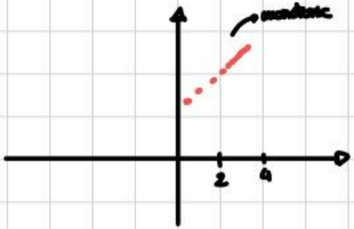
$y = \frac{1}{n}\sin n$

→ distance from the origin ρ_n
as n↑ ρ_n↓



2) $\left(\frac{3n-2}{n+1}, \left(1+\frac{1}{n}\right)^{2n}\right)_{n \in \mathbb{N}^+}$

$$\underline{P}_n \begin{cases} x = \frac{3n-2}{n+1} \\ y = \left(1+\frac{1}{n}\right)^{2n} \end{cases} \longrightarrow \text{component wise approach}$$



$\lim_{n \to \infty} x = 3$

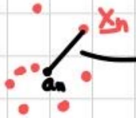
$\lim_{n \to \infty} y = e^2$

the series converges to $(3, e^2)$

DEFINITION OF CONVERGENCE

$(\underline{x}_n)_{n\in N}$ convergent sequence to $\underline{a} \in R^d$ iff $(\|\underline{x}_n - a\|)_{n\in N}$ converges to 0

$$\underline{x}_n \xrightarrow{R^d} \underline{a} \iff \|x_n - a_n\| \xrightarrow{R} 0$$



$\to \|x_n - a\| = \text{dist}(x_n, a) \in R$

$\|$

d_n sequence of real numbers

$$\lim \underline{x}_n = a$$

Theorem $\forall \varepsilon > 0 \ \exists N > 0 : \forall n > N \ |x_n - a| \leq \varepsilon$

$\downarrow$

a position after which d_n will remain smaller than ε

COMPONENT BY COMPONENT:

$\underline{x}_n$ converges to $a \iff \forall i \in \{1, 2, \dots, d\}$ the seq. $(x_{i,n})_{n\in N}$ converges to a_i

Proof:

$\Rightarrow$) $x_n \to a \Rightarrow \|x_n - a\| \to 0$ then $\sqrt{(x_1 - a_1)^2 + \dots + (x_i - a_i)^2 + \dots (x_n - a_n)^2} = 0$

$\forall i = 1, \dots, d \quad |x_{n,i} - a_i| = \sqrt{(x_{n,i} - a_i)^2} \leq \sqrt{\sum_{i=1}^{d} (x_i - a_i)^2} < \varepsilon$

possible iff $\forall i \quad x_i \to a_i$

$\Leftarrow$) $\forall\, i=1,\dots,d$ $x_{n,i} \to a_i$ then $\|x_i - a_i\| \to 0$

$$\underbrace{|x_{n,i} - a_i|}_{\text{Infinitesional}} \longrightarrow 0$$

- if you square it, it still goes to 0
- if I add

$$\sum_{i=1}^{d} |x_{n,i} - a_i|^2 \longrightarrow 0$$

then take sqrt

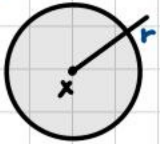
$$\sqrt{\sum_{i=1}^{d} |x_{n,i} - a_i|^2} \longrightarrow 0 \quad \Rightarrow x_n \to a_n$$

$\square$

Proposition 3.4

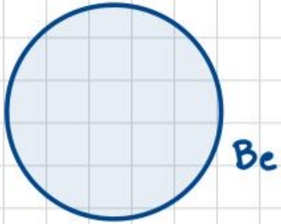
Open and Close Ball

$\dot{x} \in R^d$ center of the ball

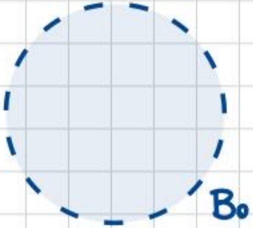


closed ball: $B_c(x,r)$ the sphere that includes the surface

$B_c(x,r) = \{y \in R^d : \|y-x\| \le r\}$



Open Ball $B_o(x,r) = \{y \in R^d : \|y-x\| < r\}$



Topologic Characterisation of the limit

$\underline{x}_n \to \underline{a}$

$\|\underline{x}_n - \underline{a}\| \to 0$

$\forall \varepsilon > 0 \ \exists M_\varepsilon : \forall n > M_\varepsilon \ \|\underline{x}_n - \underline{a}\| < \varepsilon$

$\underline{x}_n \in B_c(a, \varepsilon)$

DEF: Interior

V any set in $\mathbb{R}^d$

$\mathring{V}$ interior of V the collection of all "interior points" of the set

$\underline{P}$ is an interior point of V If $\exists\ \varepsilon>0$ and $B_c(\underline{P}, \varepsilon) \subseteq V$
(If $\exists$ a closed ball centred at $\underline{P}$ with $r=\varepsilon$ s.t. is closed and all inside V)

The collection of all interior points makes the interior of the set

Exercise

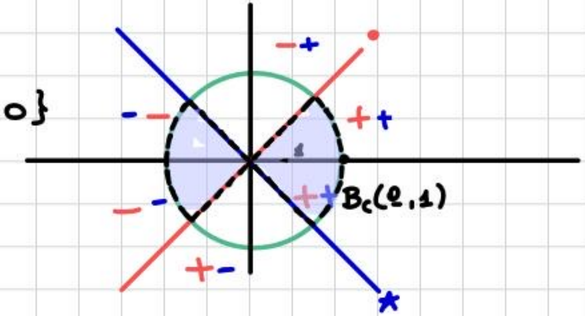
a) $A = \{ (x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1 \text{ and } x^2-y^2>0 \}$

$x^2-y^2>0$

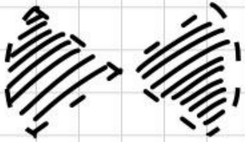
$\underbrace{(x-y)}_{I}\underbrace{(x+y)}_{II} > 0 \rightarrow$

sign I:
+ if $x>y \rightarrow \mathring{y}$

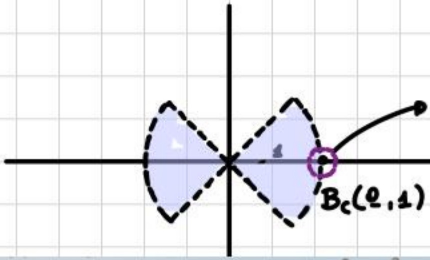
sign II:
+ if $x>-y$
$\hookrightarrow y^*$



$\mathring{A}$:



$\mathring{A}$ interior $\subseteq A$

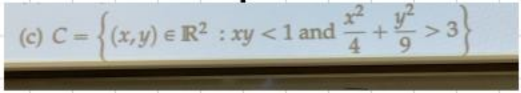


Prove this point $\overset{a}{\overset{\shortparallel}{(1,0)}}$ is not an interior point

picking $B_c(a, \varepsilon)$ and a point $(1+\frac{\varepsilon}{2}, 0) = b$ WTS $b \in B_c(a, \varepsilon)$

$\downarrow$

calculate distance $= \frac{\varepsilon}{2}$



(c) $C = \left\{(x,y) \in \mathbb{R}^2 : xy < 1 \text{ and } \frac{x^2}{4} + \frac{y^2}{9} > 3\right\}$

WTS $b \not\in A$

$\downarrow$

outside the inequality : distance of b from the origin : $\left(1+\frac{\varepsilon}{2}\right)^2 > 1$

b) $B = \{x \in R : x \in \mathbb{Q}\}$ $\quad \dot{B} = \emptyset$

Exercise 4 If I change the Ambient I change the topology

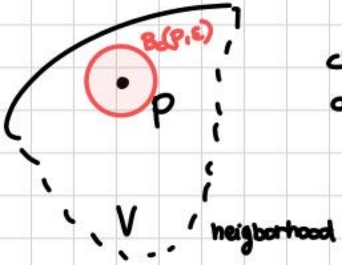
a) $E = \{x \in R : x \in [0, 10)\}$

$\mathring{E} = \{x \in R : x \in (0,10)\}$

b) $F = \{ x \in R^2 : x \in [0,10) \text{ and } y = 0\}$

$\rightarrow$ disk, since y must be $= 0$, $\mathring{F} = \emptyset$

$\mathring{F} = \emptyset$



closed and open balls are neighborhood

Neighborhood $\exists\, \varepsilon > 0$ s.t. $[\forall y \in R^d, \|x-y\| \leq \varepsilon \rightarrow y \in V]$

Theorem: Properties of Interior

- $\mathring{V} \subseteq V$
- if $V \subseteq W$ then $\mathring{V} \subseteq \mathring{W}$

  pick $x \in \mathring{V}$ i.e. $\exists\, \varepsilon > 0$ $B_c(x,\varepsilon) \subseteq V$

  $\forall y \in B_c(x,\varepsilon) \rightarrow y \in V \subseteq W$

  $B_c(x,\varepsilon) \subseteq W \rightarrow x \in \mathring{W}$

$\mathring{V} \subseteq V \subseteq \overline{V}$ ↓ closure

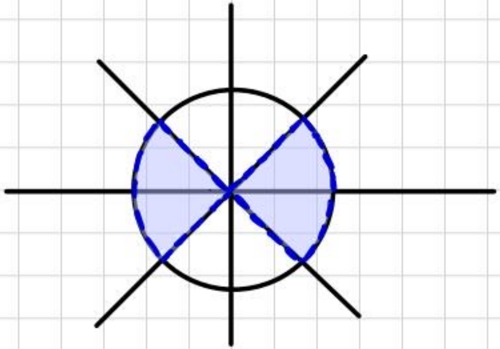
introducing STABILITY: closure

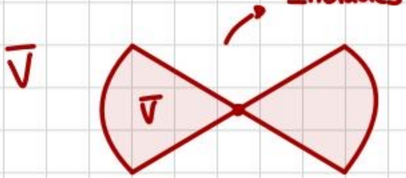
we say $x \in \overline{V}$ iff $\exists$ a sequence of elements of $\overline{V}$ converging to such x

$(y_n)_{n\in\mathbb{N}}$ s.t. $y_n \in V \ \forall n$ and $y_n \xrightarrow{\mathbb{R}^d} x$ $\quad \|y_n - n\| \longrightarrow 0.$

Remark that $y \in \overline{V}$ is not necessarely a point of $\overline{V}$.

Example 1. $V = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1 ,\ x^2 - y^2 > 0\}$





Includes also the lim of the series

$(0,0) \notin V$

$(0,0) \in \overline{V}$ in fact $\quad y_n = \left(\frac{1}{n}, 0\right) \xrightarrow{\text{coordinate wise}} (0,0)$

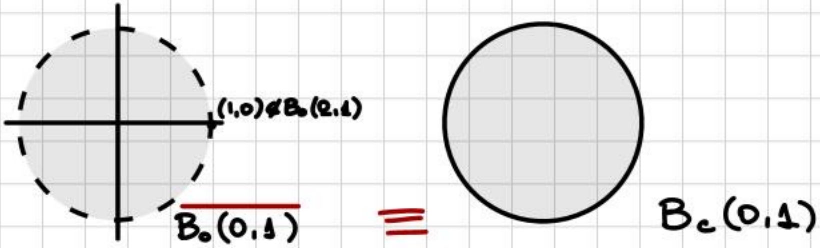
Inventing a series converging $\left(\frac{1}{2}, \frac{1}{2}\right)$ using polar coordinates

$x = \frac{1}{2}\cos\left(\frac{\pi}{2} - \frac{1}{n}\right)$

$y = \frac{1}{2}\sin\left(\frac{\pi}{2} - \frac{1}{n}\right)$

Exercise 5

Prove that $\dot{B}_c(0,1) = B_o(0,1)$ and $B_c(0,1) = \overline{B_o(0,1)}$

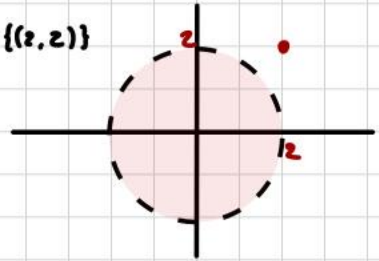


then, we have to use $\dot{B}_c(0,1) = B_o(0,1)$

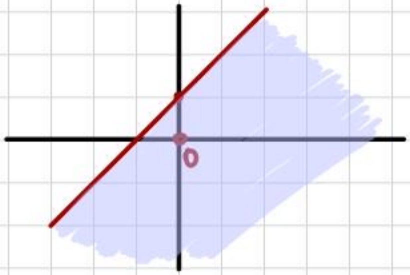
I will prove that $(1,0) \notin \dot{B}_c$ because no matter how small I pick $B((1,0),\varepsilon)$ $\exists$ a point $\left(1+\frac{\varepsilon}{2},0\right) \notin B_c$

Exercise 6

a) $A = B_o(0,2) \cup \{(2,2)\}$



b) $B = \{(x,y) \in R^2 : y-x < 1\} \setminus \{0,0\}$ $y < 1+x$



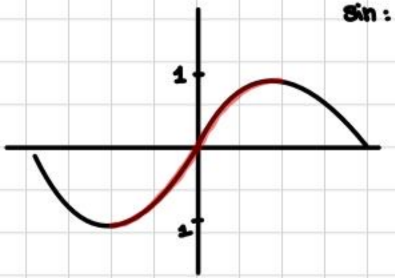
c) $C = \{x \in R : x \in Q\}$ $\cdots\cdots\to$

$\sqrt{2} \in \overline{C} = R$

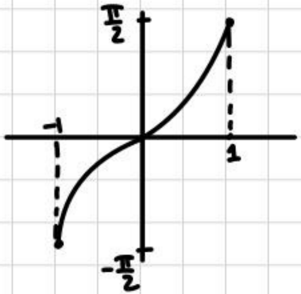
Exercise 7

$$f(x,y) = \arcsin(xy-y-2x)$$

$$f : A \subseteq \mathbb{R}^2 \xrightarrow{\text{topology}} \mathbb{R}$$
$$(x,y) \longrightarrow \arcsin(xy-y-2x)$$



$$\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1]$$
$$x \longmapsto \sin x$$



$$A = \{(x,y) \in \mathbb{R}^2 : -1 \le xy - y - 2x \le 1\}$$

find $\bar{A} =$

$$-1 \le y(x-1) - 2x \le 1$$

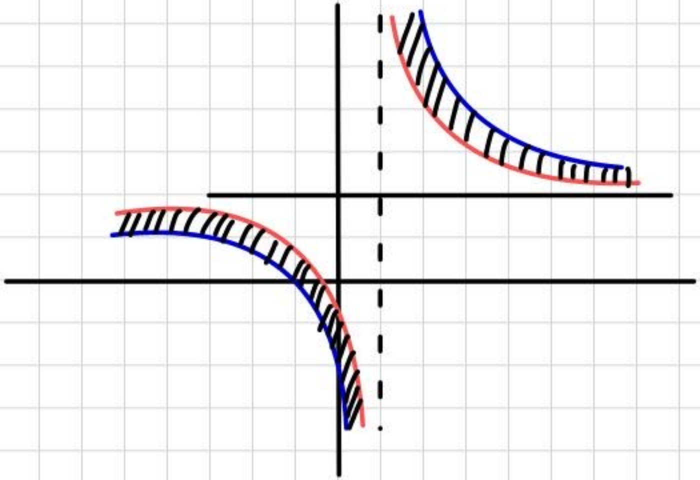
$$2x-1 \le y(x-1) \le 1+2x \qquad x = 1 \text{ not a solution}$$

if $x > 1$

$$\frac{2x-1}{x-1} \le y \le \frac{2x+1}{x-1}$$

if $x > 1$

$$\frac{1+2x}{x-1} \le y \le \frac{2x-1}{x-1}$$



Theorem: PROPERTIES OF CLOSURE

i) $V \subseteq \overline{V}$

ii) if $V \subseteq W$ then $\overline{V} \subseteq \overline{W}$

Proof: ...

STABLE UNDER THE LIMIT WITHOUT BEING INTERIOR?

Boundary: $V \subseteq \mathbb{R}^d$ we call its (topological) boundary the set $\partial V = \overline{V} \setminus \mathring{V}$

WRT a set V there will be:

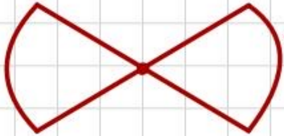
- P interior point of V if $\exists\ \varepsilon > 0$ s.t. $B_c(\underline{P}, \varepsilon) \subseteq V$

$\mathring{V} = \{\text{interior}\}$ (more than just $\underline{P} \in V$) $\mathring{V} \subseteq V$

- $\underline{P}$ exterior of V if P is an interior point of the complement $P \in (V^c)^\circ$ then $\underline{P} \in V^c$ and only surrounded by points $\in V^c$

- Boundary point (neither interior, nor exterior) !!! note. $P(\text{b.p.}) \in V$ or not $\partial V = \overline{V} \setminus \mathring{V}$

Example of before $V \cdot \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1,\ x^2 - y^2 > 0\}$



∂V:

only the surface

Thm: Link between interior and closure through complementation

$\dot{V} \cup \overline{V^c} = \mathbb{R}^d$ as a matter of fact $A \cup A^c = \text{ambient}$

$\dot{A} \cup \overline{A^c} = \text{ambient}$

If you have $\overline{(V^c)} \equiv (\dot{V})^c$ and $(\overline{V})^c \equiv (V^c)^{\circ}$

Proving: $\dot{V} \cup \overline{V}^c \equiv \mathbb{R}^d$

① $\dot{V} \cup \overline{V}^c \subseteq \mathbb{R}^d$ trivial

② $\mathbb{R}^d \subseteq \dot{V} \cup \overline{V}^c$

I pick any point of $\mathbb{R}^d$, $x \in \mathbb{R}^d$ if $x \in \dot{V}$ then the inclusion holds

else $x \notin \dot{V}$ I wts that $x \in \overline{V^c}$

$\Rightarrow \exists\, B_c(x, \varepsilon) \not\subseteq V$

so you can pick a series $B_c(x, \frac{1}{n}) \not\subseteq V$ $\Rightarrow$ I select $y_n \in B_c(x, \frac{1}{n})$ but $\notin V$

so $y_n \in V^c$

I claim $y_n \xrightarrow{\text{converges to}} x$ in $\mathbb{R}^d$

Proving convergency:

$\underbrace{\| y_n - x \|}_{\leq \frac{1}{n}} \longrightarrow 0$ so this is true : $y_n \in V^c \Rightarrow \dot{V} \cup \overline{V^c} \supseteq \mathbb{R}^d$

$\square$

Prove that $\mathring{V} \cap \overline{V^c} = \emptyset$:

By Contradiction: $\exists\, y \in \mathring{V}$ and $y \in \overline{V^c} \longrightarrow \exists$ a sequence $y_n \in V^c \;\forall n$ and $y_n \xmapsto{\mathbb{R}^d} y$

$\downarrow$ $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(y, \varepsilon) \subseteq V$

$\downarrow$ Since you have convergence $\|y_n - y\| \longmapsto 0$

so from a certain pos. $\|y_n - y\| < \varepsilon$

Contradiction

Proving : $\overline{(V^c)} = (\mathring{V})^c$

$\underset{\overline{(V^c)}}{\underset{\parallel}{\mathring{V}}} \cup \underset{(\mathring{V})^c}{\underset{\parallel}{\overline{V^c}}} = \mathbb{R}^d$ one the interior of the other

Proving: $(\overline{V})^c = (V^c)^\circ$

$V^c = W$

$V = W^c$

$(W)^\circ \cup \overline{W^c}$

DEF: Open Sets if $\mathring{V} = V$ (if every point of a set is an interior point)

Closed Sets "stable under limits" if $V = \overline{V}$

Theorem: the complement of an open set is closed, and the complement of a closed set is open

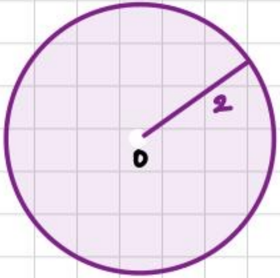
Proof: $\overline{V} = V$ closed $\Rightarrow$ $V^c \equiv (\overline{V})^c \overset{*}{=} (V^c)^\circ$ hence V^c is open

$\dot{V} = V$ open $\Rightarrow$ $V^c \equiv (\dot{V})^c \overset{*}{=} \overline{V^c}$ hence V^c is closed

$*$ Identity: $\overline{V^c} \equiv (\dot{V})^c$

Exercise 11

$B = \{(x,y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 \leq 4\}$



Find Interior boundary point

||

the origin and the surface

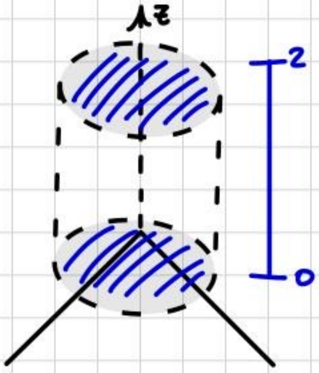
- $\partial B = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 4\} \cup \{0,0\}$

Is it open? No $\dot{B} = \{(x,y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < 4\}$

Is it close? No $\overline{B} = \{(x,y) \in \mathbb{R}^2 : 0 \leq x^2 + y^2 \leq 4\}$

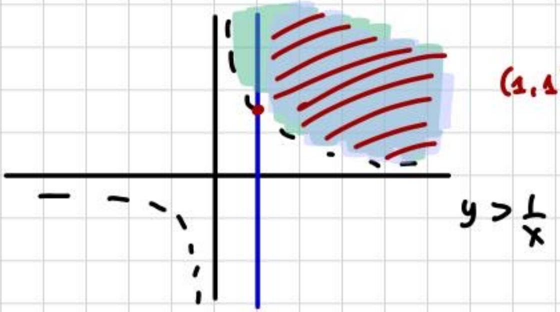
Exercise 12

$B = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 < 1 ,\ 0 \leq z \leq 2\}$



Exercise 13

$A = \{(x,y) \in \mathbb{R}^2 : xy > 1 \text{ and } x \geq 1\}$ $(1,1)$



$(1,1) \notin A$ but $\in \overline{A}$

Exercise 14

a) $\partial A = \bar{A} \setminus \mathring{A}$

b) $\overset{\circ}{\partial A} \subset \overset{\circ}{\bar{A}}$ that is the interior of ∂A is included in the interior of $\bar{A}$

$\partial A \subset \bar{A}$ by def. the interior is monotonic wrt the inclusion if $V \subseteq W \Rightarrow \mathring{V} \subseteq \mathring{W}$

c) If A is closed then ∂A has an empty interior

PROOF: Contradiction, let' say $\overline{\partial A} \neq \emptyset$ so $\exists$ $x \in (\overset{\circ}{\partial A})$ so $x \in \overset{\circ}{\bar{A}} = \mathring{A}$

but this implies $x \in (\partial A), x \in \mathring{A}$ Contradiction

$\square$

d) Example: $A = \mathring{A}$ st. ∂A has not an empty interior

$A = \mathbb{Q}$ in $\mathbb{R}$

$\partial A = \mathbb{R}$

Note $\mathbb{R}^d$ open

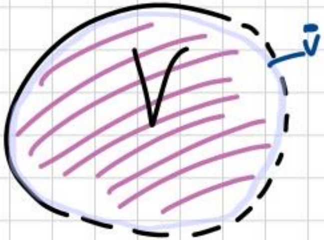
closed

Characterization of the interior and of the closure

Let V be a subset of $\mathbb{R}^d$ then,

- $\mathring{V}$ is open and it is the largest open set contained in V
- $\overline{V}$ is closed and it is the smallest closed set containing V

$$\mathring{V} \subseteq V \subseteq \overline{V}$$



PROOFS:

① proving interior is open:

$\mathring{V}$ is open $x \in \mathring{V}$ so $\exists\ \varepsilon>0$ and B_c s.t. $\underline{B_c(x,\varepsilon) \subseteq V}$ wts $B_c(x,\varepsilon) \subseteq \mathring{V}$

also

$B_o(x,\varepsilon) \subseteq V$

to pass to interior we use monotonicity :

$$\Rightarrow \left(B_o(x,\varepsilon)\right)^\circ \subseteq \mathring{V} = B_o(x,\varepsilon) \subseteq \mathring{V} \qquad \square$$

② $\mathring{V}$ is the largest open set contained in V

$A=\mathring{A}$ $\quad A \subseteq V$ then $A \subseteq \mathring{V}$

$\downarrow$ monotonicity

$A^\circ \subseteq V^\circ \Rightarrow A \subseteq V$

DEFINITIONS: Bounded Set

- V is bounded if you can find $B(\underline{x},r)$ s.t. $B(x,r) \supseteq V$

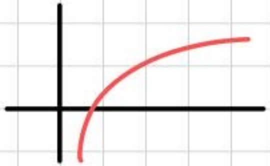
example: $f: \mathbb{R} \to \mathbb{R}$

$x \longrightarrow \sin(x$

f is boundes since its image is a bounded set in $\mathbb{R}$

but $x \to \ln x$

is unbounded

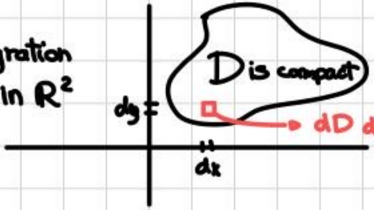


- Compact Set if at the same time is closed or bounded

INTEGRATION of functions in 2 variable (UN PICCOLO SPOILER)

$$\iint_D f(x,y)\, dD \qquad dD = dxdy$$

the domain of integration will be an AREA in $\mathbb{R}^2$



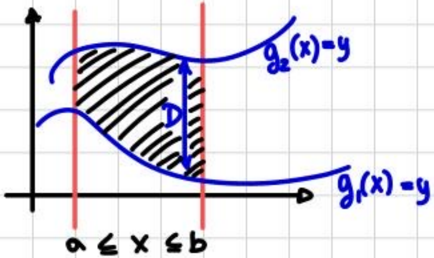
dD differential of the area $dx \cdot dy$

$$\Sigma f(x_i, y_i) \cdot (\Delta x)(\Delta y)$$

Domains of the first type: Type I

, Domains contained in a vertical stapes and close and bounded

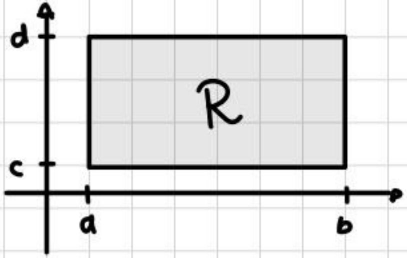
$$D = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b \text{ and } g_1(x) \le y \le g_2(x)\}$$



$$\iint_D f(x,y) = \int_a^b dx \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\, dy \right)$$

exercise 15 $D = [a,b] \times [c,d]$

a) $\iint_R (x+2y) =$



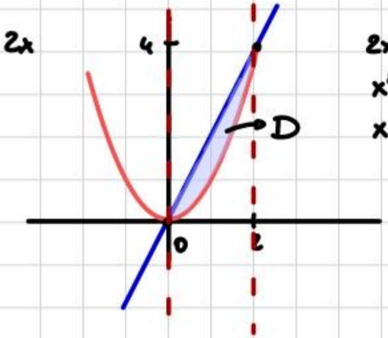
$$\int_a^b dx \int_c^d (x+2y)\,dx = \int_a^b \left(\left[xy+y^2\right]_c^d \right) dx = \int_a^b (x\cdot d + d^2 - x\cdot c - c^2)\,dx = \int_a^b \left(x(d-c) + d^2 - c^2\right) dx = \left[\frac{x^2}{2}(d-c) + d^2x - c^2x\right]_a^b$$

b) $\iint_R (xy^2) =$

the prod. allows you to move the x outside

$$\int_a^b \left(\int_c^d x\cdot y^2\,dy\right) dx = \int_a^b x \underbrace{\left(\int_c^d y^2\,dy\right)}_{\text{not included in the integration of x so...}} dx = \int_a^b x\,dx \cdot \int_c^d y^2\,dy = \left[\frac{x^2}{2}\right]_a^b \cdot \left[\frac{y^3}{3}\right]_c^d$$

exercise 16 D bounded by the curves $y=x^2$ and $y=2x$



$2x = x^2$

$x^2 - 2x = 0$

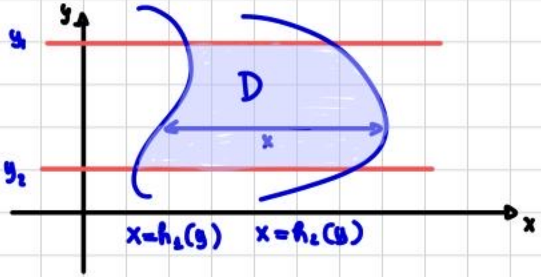
$x(x-2) = 0$

$$\iint_D (y^2+\sqrt{x})$$

$$\int_0^2 \left(\int_{x^2}^{2x} (y^2+\sqrt{x})\, dy \right) dx$$

$$\int_0^2 \left[\frac{y^3}{3} + \sqrt{x}\, y \right]_{x^2}^{2x} dx = \int_0^2 \left(\frac{8}{3} x^3 + 2x\sqrt{x} - \frac{x^6}{3} - x^2\sqrt{x} \right) dx \quad \dots$$

Domains of second type : Type 2 Horizontal slices

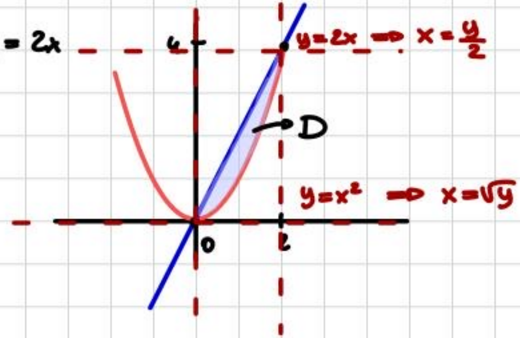


$$\int_{y_1}^{y_2}\left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right)dy$$

Example of Domain that supports both

- D bounded by the curves $y=x^2$ and $y=2x$

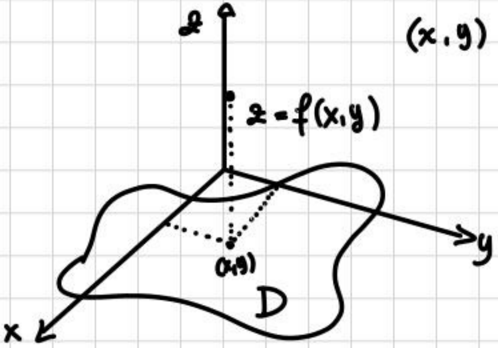
$$\iint_D (y^2+\sqrt{x})$$



Continuity and Differentiability

Functions in 2 variables : $f: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ → scalar functions

$(x,y) \longrightarrow z = f(x,y)$



M.A. 1 : $I \subseteq \mathbb{R} \to \mathbb{R}$

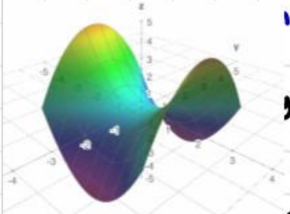
L.A. : $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ vector valued functions

domain $= \mathbb{R}^n$

$\underline{v} \longrightarrow \underline{w} = f(\underline{v})$ linear

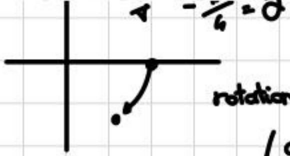
M.A. 2 : generalise $\mathbb{R}^n \to \mathbb{R}^m$

NON LINEAR



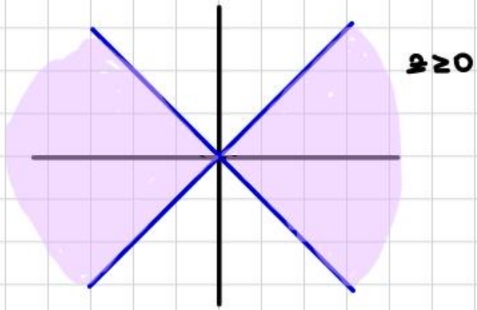
of $f(x,y) = x^2 - y^2$

$y^2 = \underbrace{(x-y)}_{I}\underbrace{(x+y)}_{II}$



rotation matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

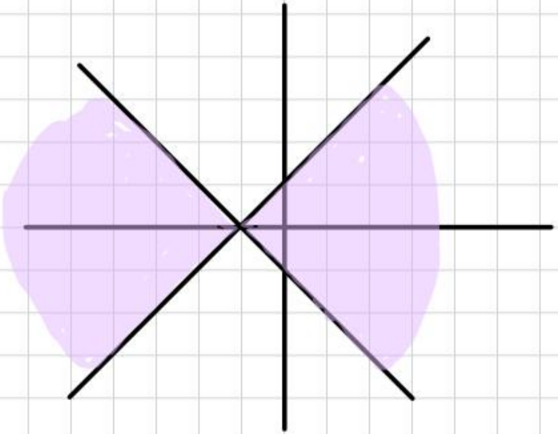


- Domain of $f(x,y) = \sqrt{|x|(x^2-(y+1)^2)}$

$\sqrt{|x|\,(x^2-(y+1)^2)}$

- $|x|$ always ≥ 0
- $(x^2-(y+1)^2) = (x+y+1)(x-y-1)$

$$>0 \quad \text{if} \quad \begin{cases} x+y+1>0; & y>-x-1 \\ x-y-1>0; & y<x-1 \end{cases}$$

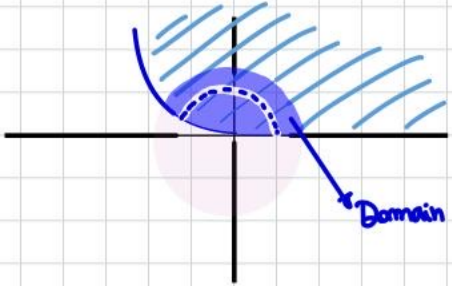


- Domain $f(x,y) = \dfrac{\sqrt{2y-x(x-|x|)}}{\ln(2-(x^2+y^2))}$

$$\begin{cases} 2y-x(x-|x|)\geq 0 \rightsquigarrow y \geq \frac{1}{2}\,x\left(x-|x|\right) \\ 2-(x^2+y^2)>0 \end{cases}$$

- if $x \geq 0$ $\quad y \geq 0$
- if $x < 0$ $\quad y \geq x^2$

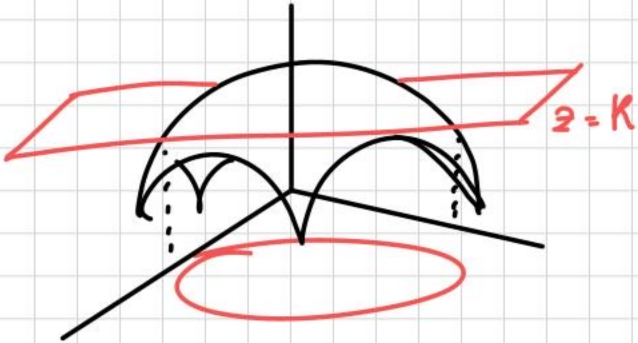
$(x^2+y^2)<2 \rightarrow B_0(0,\sqrt{2})$

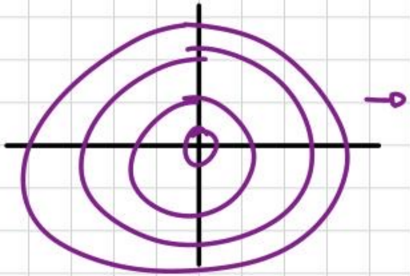


with $(x^2+y^2)\neq 1$

The k-level set $= \{(x,y) \in D_{\subseteq \mathbb{R}^2} : f(x,y) = k\}$ (sometimes it can also be a curve)

$$\begin{cases} z = f(x,y) \\ z = k \end{cases}$$

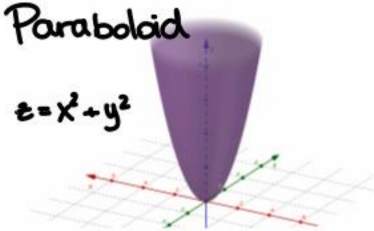




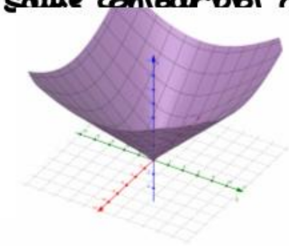
→ contour plot

Paraboloid

$z = x^2 + y^2$

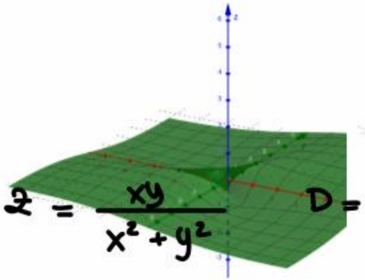


same contourplot as cone

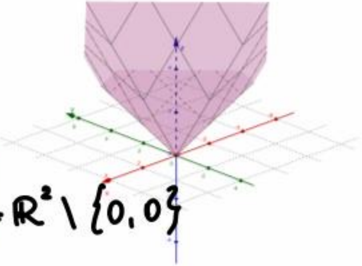


Cone

$z = \sqrt{x^2 + y^2}$

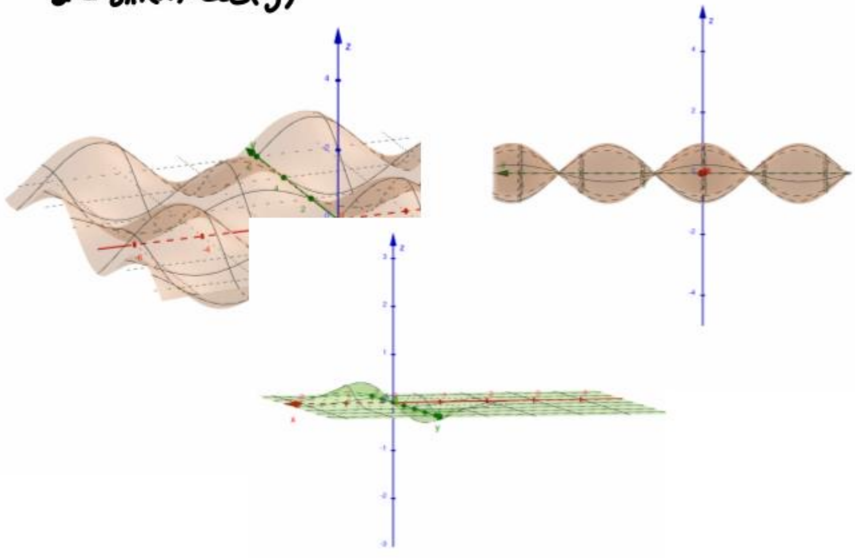


$$z = \frac{xy}{x^2 + y^2} \quad D = \mathbb{R}^2 \setminus \{0,0\}$$



$$z = |x| + |y| = \begin{cases} x + y & x, y \geq 0 \\ x - y & x \geq 0 \quad y < 0 \\ y - x & x < 0 \quad y \geq 0 \\ -x - y & x < 0 \quad y < 0 \end{cases}$$

$z = \sin(x)\cdot\cos(y)$



$z = e^{-(x^2+y^2)}\cdot x$ $D = \mathbb{R}^2$

even WRT y

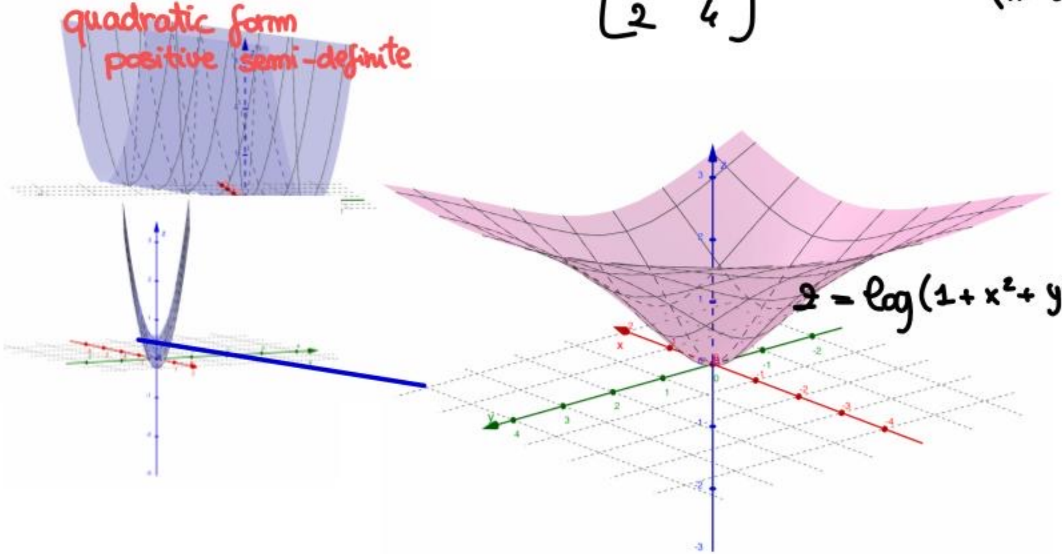
odd WRT x

Quadratic form

$$z = x^2 + 4xy + 4y^2 = (x+2y)^2$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \rightarrow \text{eigenvalues}$$

$$(x+2y)^2 = 0 \quad y = -\frac{1}{2}x$$



Limits and Continuity

$f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$

$\lim_{(x,y)\to(x_0,y_0)} f(x,y)$

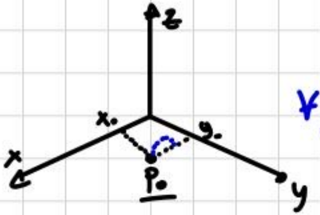
Accumulation of D limit point $\underline{x_0}$ if $\forall \varepsilon > 0$ $B_c(\underline{x_0}, \varepsilon)$ $\exists$ $y \in D \cap B_c(\underline{x_0}, \varepsilon) \setminus \{x_0\}$

(If you can build a non-costant sequence converging to x_0 made of points $\in D$)

f is continous at $\underline{x_0} \in D$ if $f(\underline{x_0}) = \lim_{(x,y)\to(x_0,y_0)} f(x,y)$
(point wise)

TRANSFERENCE PRINCIPLE

Theorem: Sequential Charachterisation of the continuity



$\forall y_n \to P_0 \Rightarrow f(y_n) \to f(P_0)$

$\Updownarrow$

f is continous at P_0

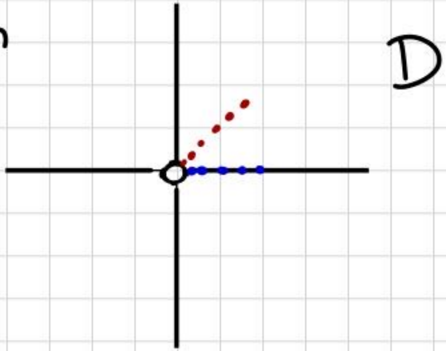
Proving a limit does not exists:

$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ we prove that the series converges to 2 different values

$\downarrow$

$\forall$

Representing Domain



$\underline{y_n} = \left(\frac{1}{n}, 0\right)$

$$f(y_n) = \frac{\frac{1}{n}\cdot 0}{\left(\frac{1}{n}\right)^2\cdot 0} = \boxed{0}$$

$\underline{y_n} = \left(\frac{1}{n}, \frac{1}{n}\right)$

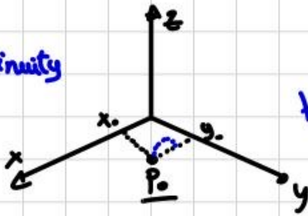
$$f(\underline{y_n}) = \frac{\frac{1}{n}\cdot\frac{1}{n}}{\frac{1}{n^2}+\frac{1}{n^2}} = \boxed{\frac{1}{2}}$$

2 series converging to 2 DIFFERENT values ⇒ the limits does not exists

⇒ $f(x,y)$ NOT CONTINOUS in $(0,0)$

TRANSFERENCE PRINCIPLE

Theorem: Sequential Charachterisation of the continuity



$$\begin{tikzcd} \forall y_n \to P_0 \Rightarrow f(y_n) \to f(P_0) \arrow[d, bend left, leftrightarrow] \\ f \text{ is continous at } P_0 \end{tikzcd}$$

Proof: $\Leftarrow$.

*Def. continuity of continuity: $\forall \varepsilon > 0\ \exists \delta > 0\ \exists y \in D\ \underset{\substack{\downarrow \\ B_c(x,\delta)}}{\|y - x\| \le \delta} \Rightarrow |f(y) - f(x)| \le \varepsilon$

By contradiction:

NOT *: $\underset{\downarrow}{\underline{\exists\, \varepsilon > 0}}\ \forall \delta > 0\ \exists y \in D\ \|y - x\| \le \delta$ AND $|f(y) - f(x)| > \varepsilon$

I select such ε which guarantees that no matter which $\delta = \frac{1}{n}$ (progressively smaller) and so $\exists\ \|y_n - x\| \le \frac{1}{n}$ AND at the same time $|f(y_n) - f(x)| > \varepsilon$

But clearly $y_n \xmapsto{R^d} x$ since $\|y_n - x\| \le \frac{1}{n} \to \|y_n - x\| \longmapsto 0$

However it cannot be that $f(y_n) \longmapsto f(x)$ since $|f(y_n) - f(x)| > \varepsilon$ Contradiction

Proof. $\Longrightarrow$

DEF:
continuity $\forall\ \varepsilon > 0\ \exists\ \delta > 0\ \forall\ y \in D\quad \underbrace{\|y - x\| \leq \delta}_{y \in B_c(x,\delta)} \quad \Longrightarrow \quad |f(y) - f(x)| < \varepsilon$

I pick $\underset{\overset{\cap}{D}}{y_n} \longmapsto x$.

$\left(\text{sooner or later } \exists N:\ \forall n > N\ y_n \in B_c(x,\delta)\right)$

this sequence will enter the ball whose existence is guaranteed from continuity

$\rightarrow |f(y_n) - f(x)| \leq \varepsilon$

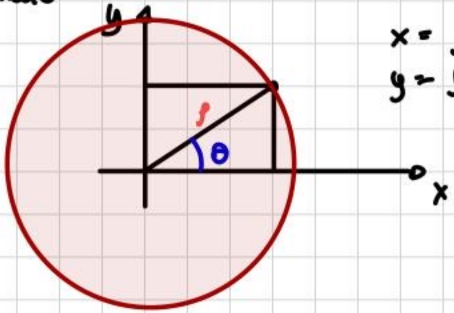
$f(y_n) \xrightarrow{R} f(x)$

□

How to prove that a limit exists & it is zero!

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

<u>idea</u> using polar coordinate



$x = \rho\cos\theta$

$y = \rho\sin\theta$

$$\Rightarrow f(x,y) = f(\rho\cos\theta, \rho\sin\theta)$$

if I find something distinct:

$$= \underbrace{H(\rho)}_{\text{(dep. on } \rho) \text{ infinitesimal}} \cdot \underbrace{G(\theta)}_{\text{(dep. on } \theta) \text{ bounded}}$$

very rarely but

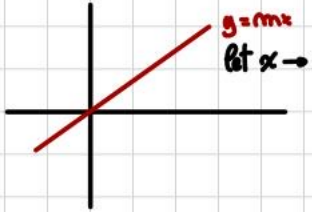
$\leq$ you can dominate it

$$\leq \text{Max amount} \cdot H(\rho) \overset{\text{goes to zero}}{=} 0$$

- $$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2}$$

$x = \rho\cos\theta$

$y = \rho\sin\theta$



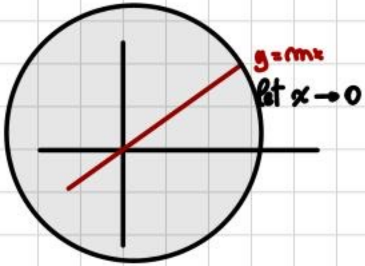
let $x \to 0$

$$f\Big|_{y=mx} = f(x, mx) = \frac{x^3 m}{x^2 + m^2 x^2} = \frac{x^3 m}{x^2(1+m)} = \frac{xm}{1+m}$$

I am calculating the limit for infinite direction

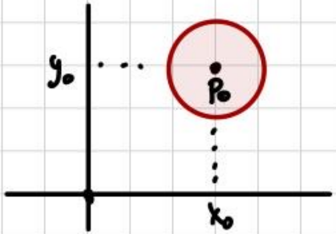
$$\lim_{x\to 0} \frac{xm}{1+m} = 0$$ BUT NOT ENOUGH to conclude $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

⇒



$$\frac{x^2y}{x^2+y^2} = \frac{\rho^2\cos^2\theta\cdot\rho\sin\theta}{\rho^2\cos^2\theta+\rho^2\sin^2\theta} = \frac{\cancel{\rho^2}\cos^2\theta\cdot\overset{\overset{y}{\|}}{\rho\sin\theta}}{\cancel{\rho^2}} \overset{y\to0}{=} 0$$

what if $$\lim_{\substack{(x,y)\to(x_0,y_0)\\ \| \\ P_0}} f(x,y)=0$$



$$\begin{cases} x=\rho\cos\theta+x_0 \\ y=\rho\cos\theta+y_0 \end{cases}$$

what if $$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=\ell \Rightarrow \lim_{(x,y)\to(x_0,y_0)} f(x,y)-\ell=0$$

$$\left|f(x_0+\rho\cos\theta,\ y_0+\rho\sin\theta)-\ell\right| \le \underbrace{H(\rho)}_{\text{infinitesimal}}\cdot\underbrace{G(\theta)}_{\text{bounded}}$$

a) $f(x,y) = \frac{x^3+y^3}{x^2+y^2}$

$$\lim_{(x,y)\to(0,0)} = \frac{\rho^3(\cos^3\theta+\sin^3\theta)}{\rho^2} = \left|\rho(\cos^3\theta+\sin^3\theta)\right| = \rho\cdot G(\theta)$$

↳ bounded

NO LIMIT: • $f(x,y) = xe^{-y/x}$

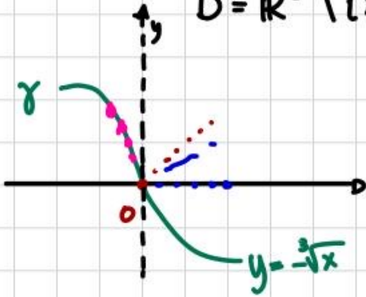
$D = \mathbb{R}^2 \setminus \{x=0\}$

$$\lim_{(x,y)\to(0,0)} xe^{-y/x}$$

$\cancel{\left(\frac{1}{n},0\right)} = \frac{1}{n} = y_n \neq f(y_n) = \cdot$

$\cancel{\left(\frac{1}{n},\ \right)} = \frac{1}{n}e^{-} = \frac{1}{en} \to 0$

$\left(\frac{1}{n}, -\sqrt[3]{\frac{1}{n}}\right)$



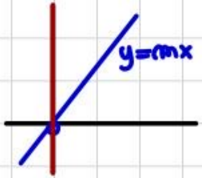
usually when you remove an entire line you can converge to the point using a curve → maybe tangent to the removed line

$f\circ\gamma = f|_\gamma$

$f(x,-\sqrt[3]{x}) = x\cdot e^{+\frac{\sqrt[3]{x}}{x}}$

$= x\cdot e^{x^{\frac{1}{3}-1}} = x\cdot e^{-\frac{2}{3}} \to \infty$

• $f(x,y) = \frac{x^3+y^3}{x^4+y^4} =$



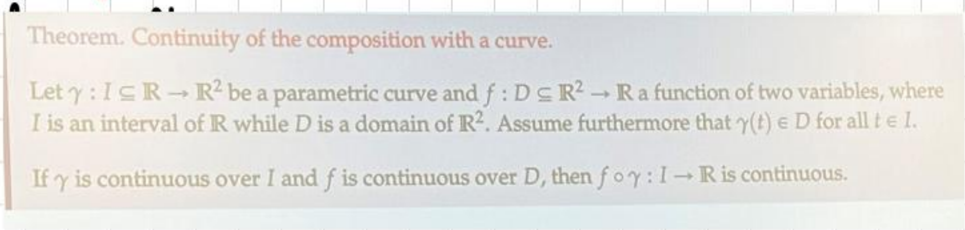
$f(x,mx) = \frac{x^{\overset{1}{\cancel{3}}}(1+m)}{\cancel{x^{2}}(1+m^4x^2)}$

$x \mapsto 0$, $f(x,mx) = 0$ ∀ straight line

now $f(\underline{0},y) = \frac{0+y^3}{0+y^4} = \frac{1}{y}$

as $y \mapsto 0$, $f(0,y) = \pm\infty$

The composition with a curve



Theorem. Continuity of the composition with a curve.

Let $\gamma : I \subseteq \mathbb{R} \to \mathbb{R}^2$ be a parametric curve and $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ a function of two variables, where I is an interval of $\mathbb{R}$ while D is a domain of $\mathbb{R}^2$. Assume furthermore that $\gamma(t) \in D$ for all $t \in I$.

If γ is continuous over I and f is continuous over D, then $f \circ \gamma : I \to \mathbb{R}$ is continuous.

Proof: using charac. of cont.

$$f \circ \gamma : I = [a,b] \longrightarrow \mathbb{R}$$

$$t \longrightarrow f(\gamma_1(t), \gamma_2(t))$$

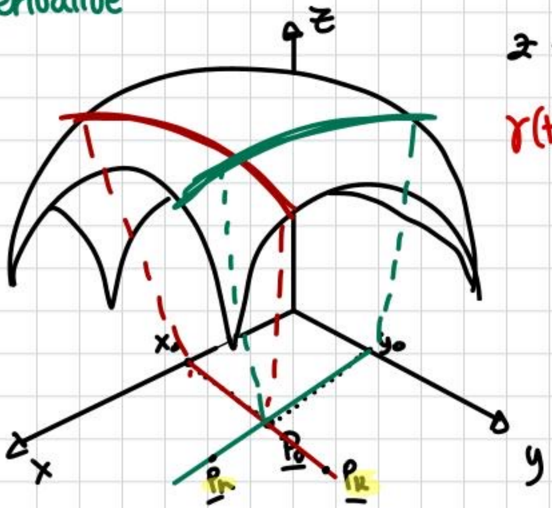
pick a sequence $t_n \to t_0 \in [a,b]$
then sequential characterisation of continuity
of $\underset{y_n}{\underset{\shortparallel}{\gamma(t_n)}} \xrightarrow{\mathbb{R}^2} \underset{y_0}{\underset{\shortparallel}{\gamma(t_0)}}$ $\to$ f is continous

$$f(\gamma_1(t_n), \gamma_2(t_n)) \longrightarrow f(\gamma_1(t_0), \gamma_2(t_0))$$

$$f(y_n) \longrightarrow f(y_0)$$

□

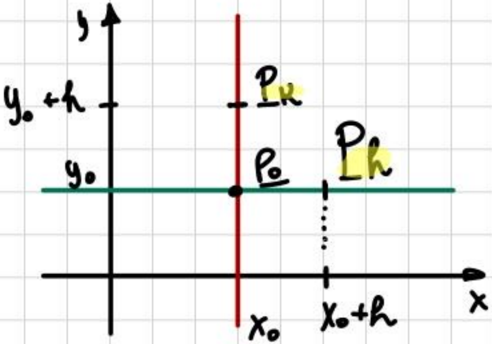
Partial Derivative



$z = f(x,y)$

$\gamma(t) = (x_0, t)$ y free

$\gamma(t) = (t, y_0)$ x free



I introduce the
different quotient along each coordinate line

partial derivative on x : $\frac{f(\underline{P_h}) - f(\underline{P_0})}{h} \to$ diff q. on x $\to \frac{\partial f}{\partial x}(P_0)$

partial derivative on y : $\frac{f(\underline{P_k}) - f(\underline{P_0})}{k} \to$ diff. q. on y $\to \frac{\partial f}{\partial y}(P_0)$

- $\lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h} = \frac{\partial f}{\partial x}(x_0, y_0)$
- $\lim_{k \to 0} \frac{f(x_0, y_0+k) - f(x_0, y_0)}{k} = \frac{\partial f}{\partial y}(x_0, y_0)$

- $f(x,y) = x^2 - y^2$ derivative at $P_0(1,2)$

$$\lim_{h\to 0} \frac{((1+h)^2 - 4) + 3}{h} = \frac{1 + 2h + h^2}{h} - \frac{1}{h} = \cancel{\frac{1}{h}} + 2 + h - \cancel{\frac{1}{h}} = 2 = \frac{\partial f}{\partial x}$$

$$\lim_{k\to 0} \frac{(1 - (2+k)^2) + 3}{k} = \frac{\cancel{1} - \cancel{4} - 4k - k^2 + \cancel{3}}{k} = -4 - k = -4 = \frac{\partial f}{\partial x}$$

- b) $\ln \frac{x-3y}{3x-2y}$

$$\frac{\partial f}{\partial x} = \frac{\cancel{3x-2y}}{x-3y} \cdot \left(\frac{(3x-2y) + (x-3y)3}{(3x-2y)^{\cancel{2}}} \right) = \frac{(3x-2y) + (x-3y)3}{(x-3y)(3x-2y)} = \frac{1}{(x-3y)} + \frac{3}{(3x-2y)}$$

$$\frac{\partial f}{\partial y} = \frac{\cancel{3x-2y}}{x-3y} \cdot \left(\frac{-3(3x-2y) + (x-3y)2}{(3x-2y)^{\cancel{2}}} \right) = \frac{-3}{x-3y} + \frac{2}{3x-2y}$$

Inverse Image of a function

If $f: D \subset R^2 \to R$ we define $f^{-1}(V)$, the inverse image of V as the subset of $x \in D$ s.t. $f(x) \in V$.

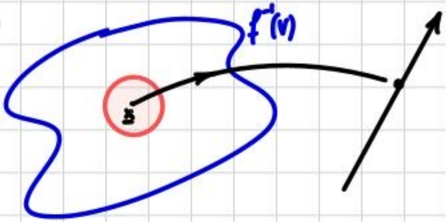
Opness is preserved when pulling back the set

THEOREM

f is continous over R^2 $\Longleftrightarrow$ $\forall\, V \subseteq R$ open, the set $f^{-1}(V)$ is open

proof

$\Longrightarrow$



V opened $z \in V$ $\exists\, B_c(z, \varepsilon) \subseteq V$

↳ tolerance $\Rightarrow \exists\, \delta$ $B(x, \delta)$

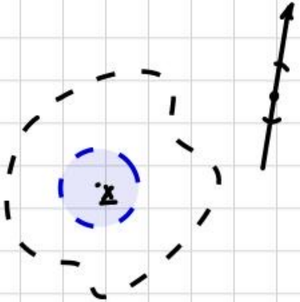
$$f(B(x,\delta)) \subseteq B_c(z, \varepsilon) \subseteq V$$

$B(x,\delta) \subseteq f^{-1}(V)$ which conclude the proof

$\Longleftarrow$ take arbitrary $x \in R^2$ $f(x) \in R$

$\forall\, \varepsilon > 0$ $B_o(f(x), \varepsilon)$

$x \in f^{-1}(B_o(f(x), \varepsilon))^{\text{open}}$



$$\boxed{f^{-1}(V^c) = \left(f^{-1}(V)\right)^c}$$

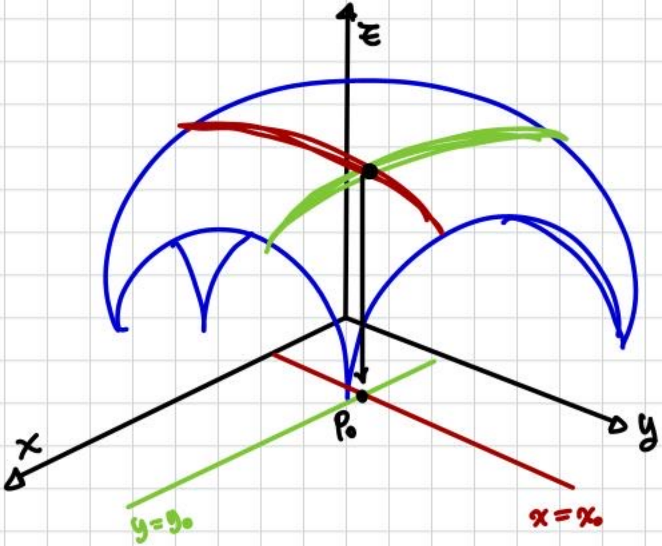
THEOREM:

f is continous over $\mathbb{R}^2$ $\Longleftrightarrow$ $\forall\ V \subseteq \mathbb{R}$ closed, the set $f^{-1}(V)$ is closed

$\Longleftarrow$ If V is open, V^c is closed and so $\left(f^{-1}(V)\right)^c$ = closed

$$\parallel$$

$f^{-1}(V^c)$ is still closed. Closure is conserved



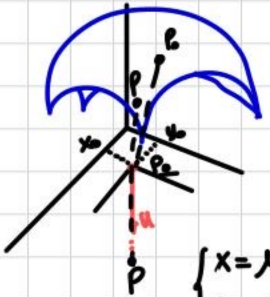
$$\boxed{f(r,y,z) = 3x^2u + ye^{2u}}$$

$$f'_x = 6xu$$

$$f'_y = e^{2u}$$

$$f'_u = 3x^2 + 2ye^{2u}$$

DIRECTIONAL DERIVATIVE



limit of the different quotient
along the direction u

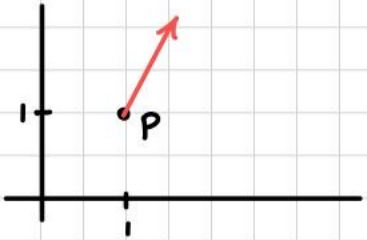
$$\lim_{t\to 0} \frac{f(x_0+\mu_1 t, y_0+\mu_2 t) - f(x_0, y_0)}{t} = \frac{\partial f}{\partial \underline{u}}(\underline{P_0})$$

$$\begin{cases} x = \mu_1 t + x_0 \\ y = \mu_2 + y_0 \end{cases}$$

!!! note $\underline{u} = \underline{e}_1$ we get $\frac{\partial f}{\partial \underline{u}} = \frac{\partial f}{\partial \underline{x}}$

$\underline{u} = \underline{e}_2$ we get $\frac{\partial f}{\partial \underline{u}} = \frac{\partial f}{\partial \underline{y}}$

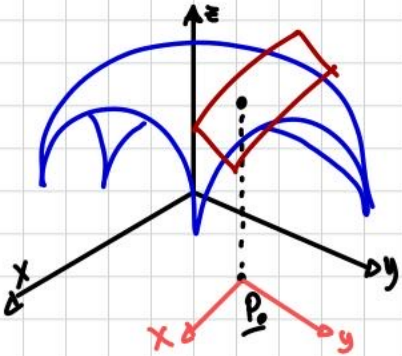
$\underline{u}\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ of the function $z = x \cdot e^{y^2}$ at $P=(1,1)$

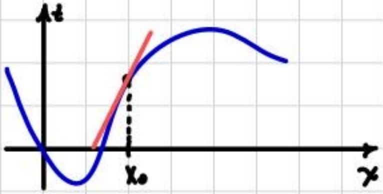


$$\lim_{h \to 0} \frac{f((1,1)+hu)-f(1,1)}{h} = \frac{\left(1+\frac{1}{2}h\right)e^{\left(1+\frac{\sqrt{3}}{2}h\right)^2} - e^2}{h} =$$

$$\lim_{h \to 0} \frac{\left(1-\frac{1}{2}h\right)e^{\left(1+\sqrt{3}h+\frac{3}{4}h^2\right)} - e^2}{h} =$$

Further regularity .. differentiabilit



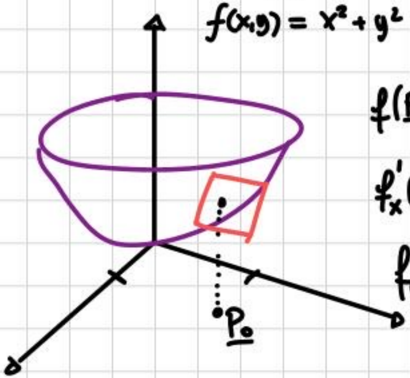


$$T_1(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$T_1(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(\underline{P_0})(x - x_0) + \frac{\partial f}{\partial y}(\underline{P_0})(y - y_0)$$

$P_0 = (1, 2)$

$f(x,y) = x^2 + y^2$



$f(\underline{P_0}) = 5$

$f'_x(\underline{P_0}) = 2$

$f'_y(\underline{P_0}) - 4$

$$z = 5 + 2(x-1) + 4(y-2)$$

hypertangent plane

$f(x,y) = xy^2 + 2yu + xyu$ at $P_0 = (1,2,-1)$

$f'_x = y^2 + yu$

$f'_y = 2xy + 2u + xu$

$f'_u = 2y + xy$

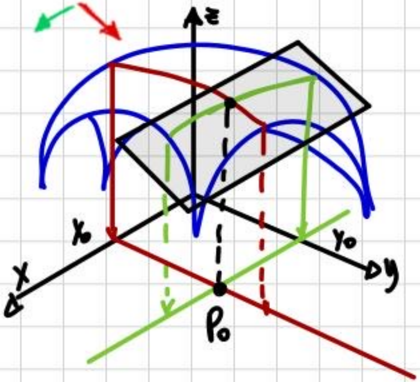
$f(P_0) = -2$

$f'_x(P_0) = 2$

$f'_y(P_0) = 1$

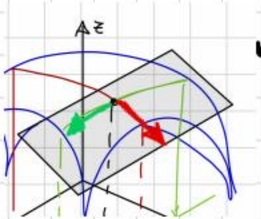
$f'_u(P_0) = 6$

$\Rightarrow z = -2(x-1) + (y-2) + 6(z+1)$



x-coordinate line $\begin{cases} x = t \\ y = y_0 \\ z = f(t, y_0) \end{cases}$

y-coordinate line $\begin{cases} x = x_0 \\ y = t \\ z = f(x_0, t) \end{cases}$



ne containing green and red arrow

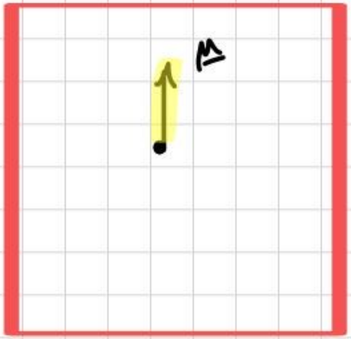
$\gamma_1' \begin{cases} 1 \\ 0 \\ f_x(P_0) \end{cases}$ $\quad \gamma_2' \begin{cases} 0 \\ 1 \\ f_y(\underline{P_0}) \end{cases}$

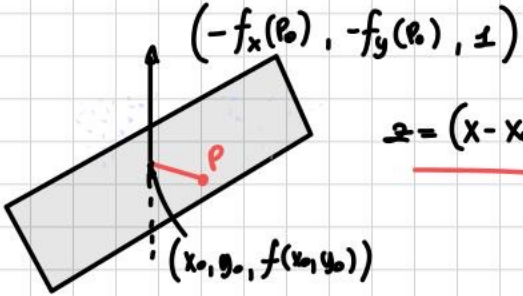
I want $\underset{(a,b,c)}{\underset{\shortparallel}{u}} \perp \gamma_1'$ and $u \perp \gamma_2'$

$$\begin{cases} a + c\, f_x(P_0) = 0 \\ b + c\, f_y(P_0) = 0 \end{cases} \qquad \underline{u} = \begin{pmatrix} -f_x(\underline{P_0}) \\ -f_y(P_0) \\ \textcircled{1} \end{pmatrix}$$

we chose to fix c

I find infinite solution depending on one parameter



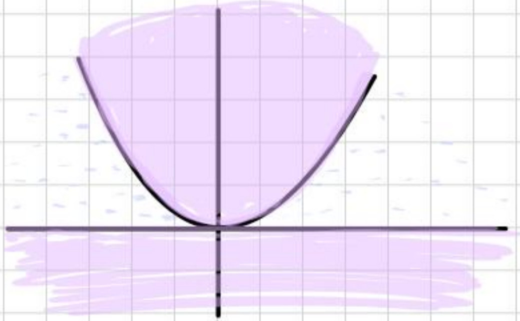


$$\underbrace{z = (x-x_0)\cdot(-f_x(P_0)) + (y-y_0)(-f_y(P_0)) + f(P_0)}_{\text{Tangent plane}}$$

If the tangent plane exists it has this equation

- $f(x,y) = \begin{cases} 0 & \text{if } 0<y<x^2 \\ 1000 & \text{elsewhere} \end{cases}$

tg exists? Directional derivatives at (0,0)



$f_x(P_0) = (1000)' = 0$

$f_y(P_0) = (1000)' = 0$

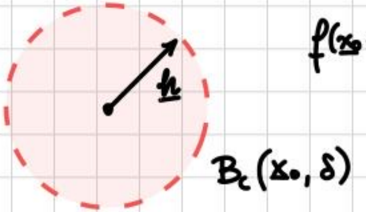
$z = f(P_0) = 1000$

$\forall\, \underline{u} \quad \frac{\partial f}{\partial u} = \lim_{t\to 0} \frac{f(0+tu_1, 0+tu_2) - f(0,0)}{t} = 0$

Continuity is not implied by the existance of the derivatives

Def. of Differentiability

$\forall \varepsilon > 0 \;\; \exists \delta > 0 \,, \; B_c(0, \delta) \;\; \forall x_0 + h \in D$



$$f(\underline{x_0} + \underline{h}) - f(\underline{x_0}) = \underbrace{f_x(x_0)h_1 + f_y(x_0)h_2}_{\text{Differential}} + o(h)$$

- f is differentiable at $\underline{P_0}$ ⇒ continuity at $\underline{P_0}$
  sufficient condition for diff.

- different condition for diff.
  $f \in C^1(D)$ D open ⇒ f is differentiable

- If 2 functions $f, g \in C^1(D)$ then $f+g \in C^1$ and $f \circ g \in C^1$. If g doesn't vanish at x_0, $f/g \in C'(D)$

Exercise

$$f(x,y) = \begin{cases} \frac{y\sin(x)}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

① Determine if f is continuous at $(0,0)$

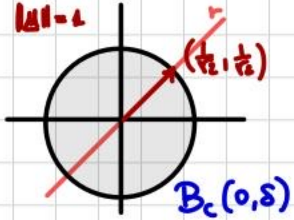
$f(0,0) = 0$

I have to dominate
$|\sin t| \leq |t|$

$\delta = \varepsilon$ because
$f(x,y)$ stays $\leq \rho$

$$\lim_{(x,y)\to(0,0)} \left(\frac{y \sin(x)}{\sqrt{x^2+y^2}} \right) \longrightarrow \lim_{\rho \to (0,0)} \frac{\cancel{\rho}\sin\theta \, \sin(\rho\cos\theta)}{\cancel{\rho}} = \left| \sin\theta \lim_{\rho \to 0} \sin(\rho\cos\theta) \right| \leq |\rho\cos\theta| \leq \rho \longrightarrow 0$$

② Calculate the directional derivative $D_v f(0,0)$ (if it exists) in the direction at $(0,0)$



Def Directional Derivative :

$$\lim_{t\to 0} \frac{f(\underline{0} + t\underline{u}) - 0}{t} = 0 = \frac{\partial f}{\partial \underline{u}}(0,0)$$

$$\lim_{t\to 0} \frac{t\frac{1}{\sqrt{2}}\sin\left(t\frac{1}{\sqrt{2}}\right)}{\sqrt{2\left(\frac{1}{\sqrt{2}}t\right)^2}} = \lim_{t\to 0} \frac{\cancel{\frac{t}{\sqrt{2}}} \cdot \frac{t}{\sqrt{2}}}{\cancel{\frac{t}{\sqrt{2}}}\sqrt{2}} = 0$$

x is free $(h,0)$
and $y=0$

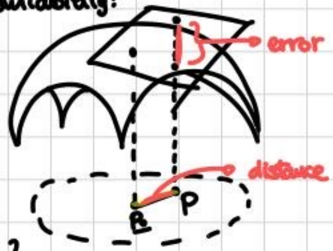
$$f_x(0,0) = \left(\frac{y\sin(x)}{\sqrt{x^2+y^2}}\right) = \frac{0-0}{h} = 0 \quad ✓$$

$$f_y(0,0) = \frac{f(0,k)-f(0,0)}{k} = \frac{0-0}{k} = 0 \quad ✓$$

$(0,k)$

Verifying differentiability:

$$\sqrt{(x-x_0)^2+(y-y_0)^2} = \mathrm{dist}(P_0,P)$$



Now I not only move down or up. I move in the ball

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - \overbrace{0}^{T_2(0,0)}}{\sqrt{h^2+k^2}} = o(h) \neq \emptyset \xrightarrow{\text{polar coordinates}} \lim_{\rho\to 0} \frac{f(\rho\cos\theta, \rho\sin\theta)}{\rho \cdot \|u\|} = \frac{\rho\sin\theta \cdot \overset{\to 0}{\sin(\rho\cos\theta)}}{\rho^2} = ?$$

↳ norm of the increment

Finding two functions that converge to two different limits

- $f\left(\frac{1}{n},\frac{1}{n}\right) = f(x_n) \longrightarrow \dfrac{\frac{1}{n}\sin\left(\frac{1}{n}\right)}{\frac{\sqrt{2}}{n}} = \dfrac{\frac{1}{n^2}}{\frac{\sqrt{2}}{n}} = \frac{\sqrt{2}}{n}$

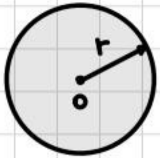
Differentiability Second Def.

$L : \mathbb{R}^2 \to \mathbb{R}$
$\underline{h} \to L(h) = ah_1 + bh_2$

f is differentiable $\Longleftrightarrow$ $\exists\, r > 0$ and a l. applic. $L : \mathbb{R}^2 \to \mathbb{R}$ s.t. for $h \in B_2(0,r)$ there holds $x_0 + h \in D$ and $f(x_0+h) = f(x_0) + L(h) + o(h)$

In this case the partial derivatives exists and $L(h,k) = h f_x(x_0) + k f_y(x_0)$

$\Leftarrow$) To prove differentiability we need existance of f_x, f_y and continuity.



we know $f(x_0+h,y_0) - f(x_0,y_0) = L(h,0) + o(h)$

$$= \frac{h\,L(1,0) + o(h)}{h} \quad \text{linearity}$$

$$= L(1,0) + \frac{o(h)}{h} = o(1)$$

Thus, passing to the limit

$$f_x(x_0) = L(1,0) = L(e_1)$$

$$f_y(x_0) = L(0,1) = L(e_2)$$

thus $L(h,k) = h\,L(1,0) + k\,L(0,1)$
$L(h,k) = h\,f_x(x_0) + k\,f_y(x_0)$

$\square$

The Chain Rule:

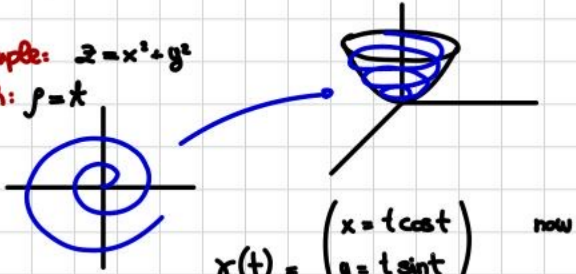
$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))\cdot g'(x)$$

$$\frac{d}{dt} f\left(x(t), y(t)\right) = f_x\left(\gamma(t)\right)\cdot x'(t) + f_y\left(\gamma(t)\right)\cdot y'(t) = \nabla f\left(\gamma(t_0)\right)\cdot \gamma'(t_0)$$

$$f\left(\gamma(t)\right)$$

example: $z = x^2 + y^2$

path: $\rho = t$



$$\gamma(t) = \begin{pmatrix} x = t\cos t \\ y = t\sin t \end{pmatrix}$$

now $f(t\cos t, t\sin t) = (t\cos t)^2 + (t\sin t)^2 = t^2$

$$\frac{\partial}{\partial t} f\left(\gamma(t)\right) \overset{\text{chain rule}}{=} \underset{\begin{pmatrix} 2x \\ 2y \end{pmatrix}}{\underset{\downarrow}{\nabla f(\gamma(t))}} \bullet \gamma'(t) = \begin{pmatrix} 2t\cos t \\ 2t\sin t \end{pmatrix} \bullet \begin{pmatrix} \cos t - t\sin t \\ \sin t + t\cos t \end{pmatrix}$$

$$= 2t\cos^2 t - \cancel{2t^2\sin t\cos t} + 2t\sin^2 t + \cancel{2t^2\cos t\sin t} = 2t$$

A particular case of the gradient rule : The Gradient Rule

to calculate directional derivative : a l.c. of partial derivatives with $\underset{\text{components of } \underline{u}}{\underset{\shortparallel}{\text{coeff.}}}$

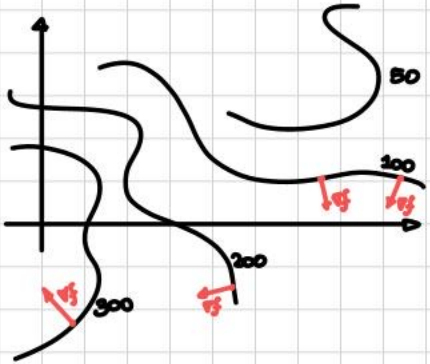
$$\frac{\partial f}{\partial \underline{u}} = \nabla f \cdot \underline{u}$$

$$\frac{\partial f}{\partial \underline{u}} = \mu_1 f_x(\underline{x}_0) + \mu_2 f_y(\underline{x}_0)$$

example: $z = x^2 + y^2$
$P = (1, 2)$
$\underline{u} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

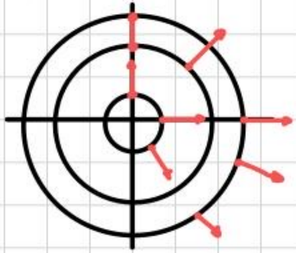
$$\frac{\partial f}{\partial \underline{u}} = \frac{1}{2} f_x(1,2) + \frac{\sqrt{3}}{2} f_y(1,2) = 1 + 2\sqrt{3}$$

Orthogonality of the gradient with level sets



Gradient always orthogonal
to the level sets

$z = x^2+y^2$ as level sets



Vector Valued functions

$$\begin{tikzcd} f: \overset{\mathbb{R}^p}{V} \arrow[r] & W^{\cdot \mathbb{R}^q} \\ \underline{v} \arrow[r, mapsto] & \underline{w} = f(\underline{v}) \end{tikzcd}$$

↳ functions that have a multioutputs

Conversely to Linear Algebra
we ~~won't~~ have the assumption
of linearity ⟹ NO REP. MATRIX
$f(\underline{v}) \neq A \cdot \underline{w}$

- Vector Valued function (non-linear)

geometric meaning:

DEFINITION: A function on $f: D \subseteq \mathbb{R}^d \to \mathbb{R}^p$ is the data of $D \subseteq \mathbb{R}^d$ which is the domain of the definition, and f which associates to every point x of D a vector $f(x) \in \mathbb{R}^p$.

$$\begin{tikzcd} f: D \subseteq \mathbb{R}^d \arrow[r] & \mathbb{R}^p \\ \underline{v} \arrow[r, mapsto] & \underline{w} = f(\underline{v}) = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \begin{matrix} = x_1 \cdot x_2^2 + \sin x_1 \cdot x_3 - \sqrt{x_3 \cdot x_p} \\ = \text{-----} \\ = \text{- - - -} \end{matrix} \end{tikzcd}$$

Graph of a function:

Let $f: D \subseteq \mathbb{R}^d \to \mathbb{R}^p$ be a vector valued function of d variables. Its graph is the subset of $\mathbb{R}^{d+p}$ made of the points (x, y) s.t. $x \in D \subseteq \mathbb{R}^d$, $y \in \mathbb{R}^p$ and $y = f(x)$.

Image of a function

▷ The vector valued functions are → vector field

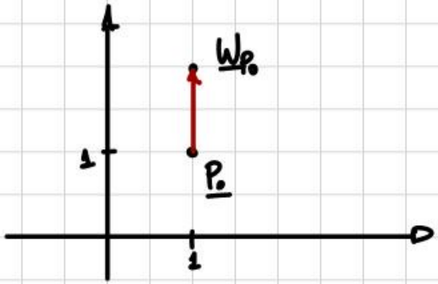
if $p = q$ (endomorphism) I can imagine to talk abt vector field

ex. $2 = p = q$

$$f : R^2 \longrightarrow \overset{\text{endomorphism}}{R^2}$$

$$\underline{v} = (x,y) \longmapsto \underline{w} = (x \cdot y \, , x^2 + y^2)$$

$\forall \, \underline{P_0} \longmapsto \underline{W}_{P_0}$ I associate a direction to every point of the domain



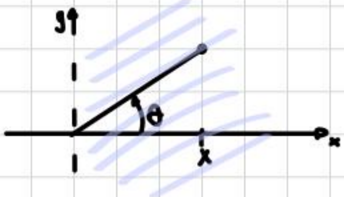
• change of coordinates in endomorphism

$$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \overset{=A}{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix}$$

from cartesian to polar

$$f : R^2 \longrightarrow R^2$$

$$(x,y) \longrightarrow (\rho,\theta) = \begin{cases} \rho = \sqrt{x^2+y^2} = w_1 = f_1(x,y) \\ \theta = \arctan \frac{y}{x} = w_2 = f_2(x,y) \end{cases}$$



from polar to cartesian

$$f^{-1} = \begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta \end{cases}$$

Parametric Curves are vector valued functions $I \subseteq R \longrightarrow R^3$

$t \longmapsto \gamma(t) = \left(f_1(t), f_2(t), f_3(t)\right)$

$p = 1 \quad q > 1$ → this are parametric curves

(Not only in R^1, also in R^2)

Parametric Surfaces are vector valued functions $A \subseteq R^2 \longrightarrow R^3$

$$(u,v) \longmapsto \begin{cases} x = f_1(u,v) \\ y = f_2(u,v) \\ z = f_3(u,v) \end{cases}$$

The Sphere

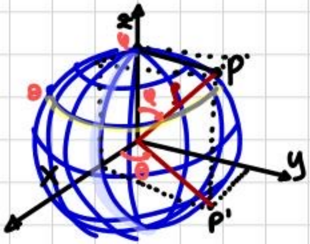
the sphere of radius R in R^3

Way of representing

- $x^2 + y^2 + z^2 = R^2$ (level set of a given function in 1 variable)
- $z = \sqrt{R^2 - x^2 - y^2}$ half of it (graph of a function in 2 variables)
- spherical coordinates

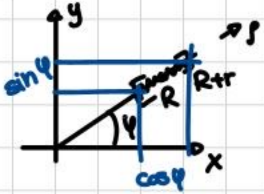
  Parametric Surface

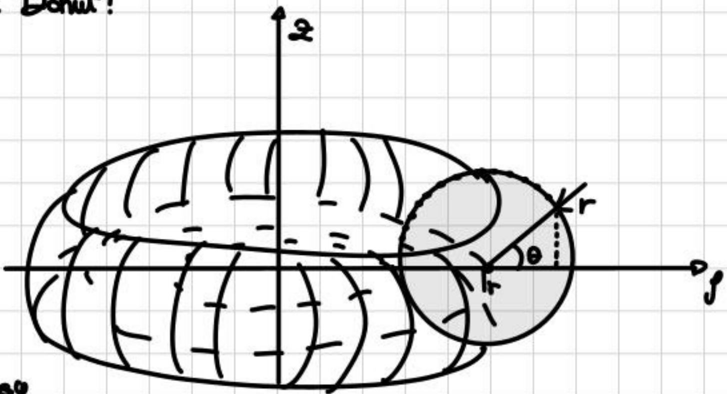
  (θ, φ) 2 components



$$\begin{cases} x = \overset{R}{\rho}\sin\varphi\cos\theta \\ y = \overset{R}{\rho}\sin\varphi\sin\theta \\ z = \overset{R}{\rho}\cos\varphi \end{cases}$$

Try to parametrize the Donut!





$\rho = R + R\cos\theta$

$\downarrow$

$$\begin{cases} x = (R + r\cos\theta)\cos\varphi \\ y = (R + r\cos\theta)\sin\varphi \\ z = r\sin\theta \end{cases}$$

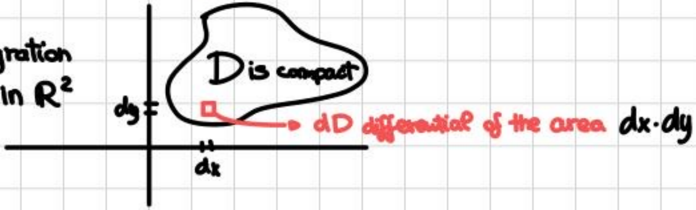
$\theta \in [0, 2\pi]$

$\varphi \in [0, 2\pi)$

INTEGRATION of functions in 2 variable (UN PICCOLO SPOILER)

$$\iint_D f(x,y)\, \underset{dxdy}{\underset{\|}{dD}}$$

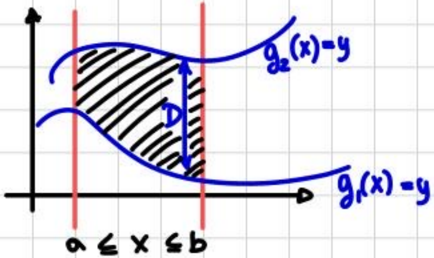
the domain of integration will be an AREA in $\mathbb{R}^2$



$$\Sigma f(x_i,y_i)\cdot(\Delta x)(\Delta y)$$

Domains of the first type: Type I , Domains contained in a vertical stripes and close and bounded

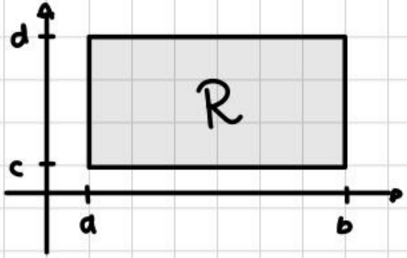
$$D=\{(x,y)\in\mathbb{R}^2 \mid a\le x\le b \text{ and } g_1(x)\le y\le g_2(x)\}$$



$$\iint_D f(x,y) = \int_a^b dx \left(\int_{g_2(x)}^{g_2(x)} f(x,y)\,dy\right)$$

exercise 15 $D = [a,b] \times [c,d]$

a) $\iint_R (x+2y) =$



$$\int_a^b dx \int_c^d (x+2y)\,dx = \int_a^b \left(\left[xy+y^2\right]_c^d\right) dx = \int_a^b (x\cdot d + d^2 - x\cdot c - c^2)\,dx = \int_a^b \left(x(d-c) + d^2 - c^2\right) dx = \left[\frac{x^2}{2}(d-c) + d^2x - c^2x\right]_a^b$$

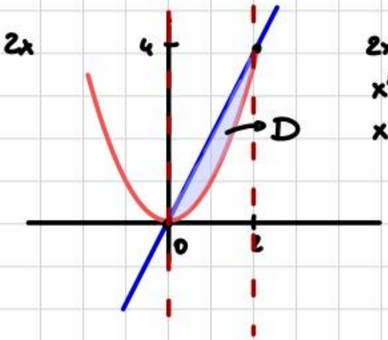
b) $\iint_R (xy^2) =$

the prod. allows you to move the x outside

$$\int_a^b \left(\int_c^d x\cdot y^2\,dy\right) dx = \int_a^b x\left(\underbrace{\int_c^d y^2\,dy}\right) dx = \int_a^b x\,dx \cdot \int_c^d y^2\,dy = \left[\frac{x^2}{2}\right]_a^b \cdot \left[\frac{y^3}{3}\right]_c^d$$

not included in the integration of x so...

exercise 16 D bounded by the curves $y=x^2$ and $y=2x$



$2x = x^2$

$x^2 - 2x = 0$

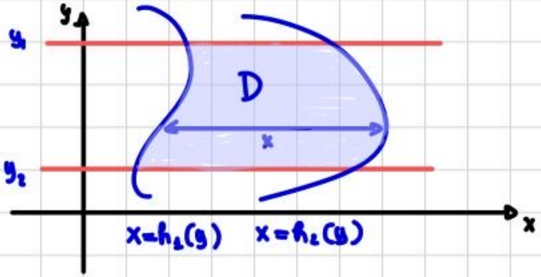
$x(x-2)=0$

$$\iint_D (y^2+\sqrt{x})$$

$$\int_0^2 \left(\int_{x^2}^{2x} (y^2+\sqrt{x})\,dy \right) dx$$

$$\int_0^2 \left[\frac{y^3}{3} + \sqrt{x}\,y \right]_{x^2}^{2x} dx = \int_0^2 \left(\frac{8}{3}x^3 + 2x\sqrt{x} - \frac{x^6}{3} - x^2\sqrt{x} \right) dx \quad \dots$$

Domains of second type : Type 2 Horizontal slices

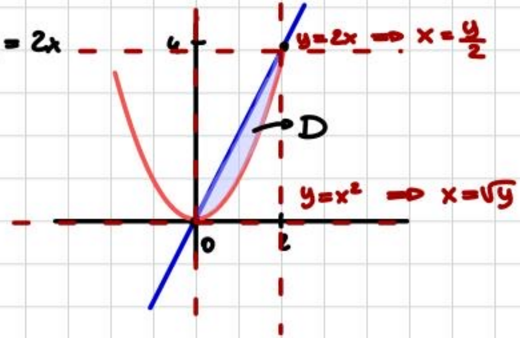


$$\int_{y_1}^{y_2}\left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right)dy$$

Example of Domain that supports both

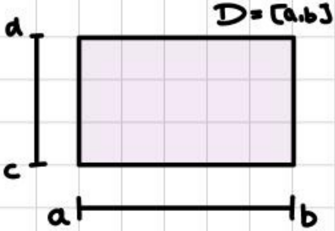
- D bounded by the curves $y=x^2$ and $y=2x$

$$\iint_D (y^2+\sqrt{x})$$



The Fubini Theorem → for a Domain which is a rectangle

$D = [a,b] \times [c,d]$



$$\iint_D f(x,y) = \int_a^b \left(\int_c^d \boxed{f(x,y)}\, dy \right) dx = \int_c^d \left(\int_a^b \boxed{f(x,y)}\, dx \right) dy$$

both types!

f is continuous

$$\Rightarrow f(x,y) = X(x) \cdot Y(y)$$

$$\iint_D f(x,y) = \int_a^b \left(\int_c^d X(x) \cdot Y(y)\, dy \right) dx = \int_a^b X(x)\, dx \int_c^d Y(y)\, dy$$

Corollary: the double integral
can be seen as a product of two functions in their respective domain (Very Lucky Case!)

Fubini for more general domains

Exercise 1

a) $D: \{ 1 \leq x \leq 2 \text{ and } 1 \leq y \leq 3 \}$

$$\iint_D xye^{x+y}\,dx\,dy = \iint_D xe^x \cdot ye^y\,dx\,dy = \int_1^2 xe^x\,dx \int_1^3 ye^y\,dy = \left[e^x(x-1)\right]_1^2 \left[e^y(y-1)\right]_1^3 = e^2 \cdot e^3 = e^5$$

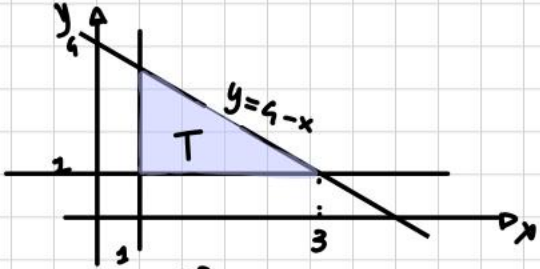
$$\downarrow$$

$$xe^x - \int e^x\,dx = e^x(x-1) + e$$

b) $T: \{ x \geq 1, \; y \geq 1 \text{ and } x+y \leq 4 \}$

First: Understanding the shape of the domain

$$\iint_T \frac{1}{(x+y)^4}\,dx\,dy = \int_1^3 \left(\int_1^{4-x} \frac{1}{(x+y)^4}\,dy \right) dx$$



① $\displaystyle \int_1^{4-x} \frac{1}{(x+y)^4}\,dy = \frac{1}{-3}\left[(x+y)^{-4+1}\right]_{y=1}^{y=4-x} = \frac{4^{-3}}{-3} - \frac{1}{-3}(x+1)^{-3}$

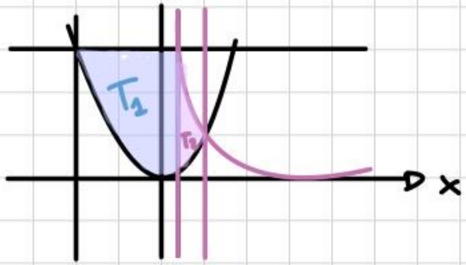
② $\displaystyle \frac{1}{3}\int_1^3 \left(4^{-3} + (x+1)^{-3}\right) dx = \frac{1}{3}\left[4^{-3}x + \frac{1}{-2}(x+1)^{-2}\right]_1^3$

Exercise 2

$T_1 = \{ -2 \leq x \leq \frac{1}{4} \text{ and } x^2 \leq y \leq 4 \}$

$T_2 = \{ \frac{1}{4} \leq x \leq 1 \text{ and } x^2 \leq y \leq \frac{1}{x} \}$

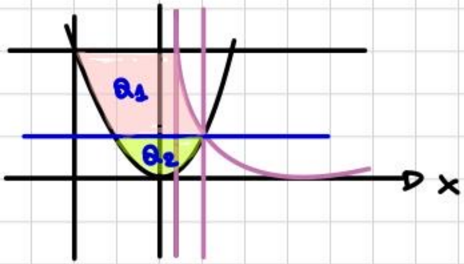
$T_1 \cup T_2$:



length of $(a,b) = \int_a^b 1 \, dt$

area of $(T_1) = \iint_{T_1} 1 = \int_{-2}^{\frac{1}{4}} \left(\int_{x^2}^{4} 1 \, dy \right) dx = \int_{-2}^{\frac{1}{4}} (4 - x^2) \, dx = \left[4x - \frac{x^3}{3} \right]_{-2}^{\frac{1}{4}} = 1 - \frac{1}{4^3 \cdot 3} - 8 + \frac{8}{3}$

area of $(T_2) = \iint_{T_2} 1 = \int_{\frac{1}{4}}^{1} \left(\int_{x^2}^{\frac{1}{x^2}} 1 \, dy \right) dx$



area of $(T_1 \cup T_2)$ = area $(Q_1 \cup Q_2)$ I divide in a different way

- $Q_1 = \{ (x,y) \in R^2 : 1 \leq y \leq 4 , -\sqrt{y} \leq x \leq \frac{1}{y} \}$
- $Q_2 = \{ (x,y) \in R^2 : 0 \leq y \leq 1 , -\sqrt{y} \leq x \leq +\sqrt{y} \}$

$\Rightarrow$ area $(Q_1 + Q_2)$

$$\nearrow y^{\frac{1}{2}} = y^{\frac{3}{2}} \cdot \frac{2}{3}$$

$$\text{area}(Q_2) = \iint_{Q_2} 1 = \int_0^1 \left(\int_{-\sqrt{y}}^{+\sqrt{y}} dx \right) dy = \int_0^1 2\sqrt{y} \, dy = 2\int_0^1 \sqrt{y} \, dy = \left[\frac{4}{3} y^{\frac{3}{2}} \right]_0^1 = \frac{4}{3}$$

Quadratic Forms

Quadratic Form is a sum of monomials of degree 2

1 variable $f(x) = ax^2$

2 variables $f(x,y) = ax^2 + by^2 + cx\cdot y$

3 variables $f(x,y,z) = ax^2 + by^2 + cz^2 + dx\cdot y + ex\cdot z + fy\cdot z$

I can associate a Matrix

$ax^2 + by^2 + cxy$

$$\begin{bmatrix} a & \frac{c}{2} \\ \frac{c}{2} & b \end{bmatrix}$$

coeff. of the square

Now I can rewrite:

$$\underline{v} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad A\cdot\underline{v} = \begin{bmatrix} a & \frac{c}{2} \\ \frac{c}{2} & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + \frac{c}{2}y \\ \frac{c}{2}x + by \end{bmatrix}$$

$$\underline{v} \bullet A\underline{v} = ax^2 + \frac{c}{2}xy + \frac{c}{2}xy + by^2 = ax^2 + by^2 + cxy$$

Every quadratic form can be written as a symmetric matrix A s.t.

$$f(\underline{x}) = \underline{x}\cdot A\cdot\underline{x}$$

$\Downarrow$

$f(\underline{0}) = 0$ Always

Quadratic Forms are classified in terms of their SIGN:

- positive definite
  if $f(x) > 0 \quad \forall x \neq 0 \in \mathbb{R}^n$
- negative definite
  if $f(x) < 0 \quad \forall x \neq 0 \in \mathbb{R}^n$
- indefinite
  if $f(x)$ attains values of opposite sign
  $\Rightarrow \exists\, x^*$ and x° s.t. $f(x^*) < 0 < f(x^\circ)$
- positive semidefinite
  if $f(x) \geq 0 \quad \forall x \in \mathbb{R}^n$
- negative semidefinite
  if $f(x) \leq 0 \quad \forall x \in \mathbb{R}^n$

Examples

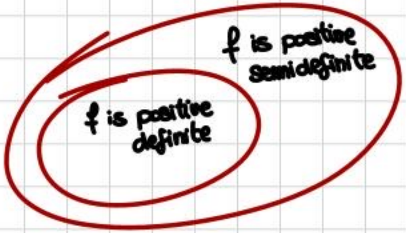
positive semi-definite : $f(x,y) = (x-y)^2$

positive definite : $f(x,y) = x^2 + y^2$

negative semi-definite : $f(x,y) = -(x-y)^2$

negative definite : $-x^2 - y^2$

indefinite : $x^2 - y^2$ OR xy



Spectral theorem:

the sign of the eigenvalues gives the sign of the quadratic form.

Even there are other elements apart from the Diagonal $\begin{bmatrix} * & & \square \\ \square & * & * \\ & & * \end{bmatrix}$ → the sign is less predictable but still depends on eigenvalues

Given H a symmetric Matrix with real coefficients. Then H is Diagonalizable in an Orthonormal basis $\Bigg\}$ $\exists$ B(invertible, orthogonal) s.t. $B^T \cdot H \cdot B = \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}$

$B^T \cdot B = I$

What does it mean orthogonal matrix?

$$\underbrace{\begin{bmatrix} \cdots & v_1 & \cdots \\ \cdots & v_2 & \cdots \\ & \vdots & \\ \cdots & v_n & \cdots \end{bmatrix}}_{H} \cdot \underbrace{\begin{bmatrix} \vdots & \vdots & & \vdots \\ v_1 & v_2 & \cdots & v_n \\ \vdots & \vdots & & \vdots \end{bmatrix}}_{H^T} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \; 1 \end{bmatrix}$$

$v_1, \dots, v_n$ orthogonal and unitary basis of $R^n \Rightarrow \begin{cases} v_i \cdot v_j = 0 & \text{for } i \neq j \\ v_i \cdot v_j = 1 & \text{for } i = j \end{cases}$

Then: the sign of a quadratic form depends on the sign of the eigenvalues

Theorem:

Consider $A = A^T$

★ A is positive definite $\Leftrightarrow$ its eigenvalues are all >0

H A is positive semidefinite $\Leftrightarrow$ its eigenvalues are all ≥ 0

Proof:

★ $\Rightarrow$ I Know that $f(\underline{x}) = \underline{x} \cdot A\underline{x} > 0 \quad \forall \underline{x} \neq \underline{0}$

pick $\underline{v}_i$ unitary eigenvector $\qquad \underline{v}_i \neq 0 \quad A\underline{v}_i = \lambda_i \underline{v}_i$

$$0 < f(\underline{v}_i) = \underline{v}_i \cdot A\underline{v}_i = \underline{v}_i \cdot \lambda_i \underline{v}_i = \lambda_i \underline{v}_i \cdot \underline{v}_i = \lambda_i \qquad \Rightarrow \qquad \lambda_i > 0$$

$\Leftarrow$ I Know that $\lambda_i > 0$

$$f(x) = \underline{x} \cdot A\,\underline{x}$$

$$\hookrightarrow A = A^T$$

$$\Rightarrow \exists\, B \text{ s.t. } B^T \cdot A \cdot B = \Lambda$$

multiplying $B \cdot B^T \cdot A \cdot B \cdot B^T = B\,\Lambda\,B^T \Rightarrow A = B\Lambda B^T$

$$\Rightarrow \underline{x} \cdot A\underline{x} = \underbrace{\boxed{\underline{x}^T \cdot B}}_{\underline{y}^T} \cdot \Lambda \cdot \underbrace{\boxed{B^T \cdot \underline{x}}}_{\underline{y} = B^T \cdot \underline{x}} = \underline{y}^T \cdot \Lambda\, \underline{y} = \sum_{i=1}^{n} \lambda_i \cdot y_i^2 > 0$$

$\square$

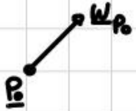
H $\Rightarrow$ Same proceeding but including 0 : $f(\underline{x}) = \underline{x} \cdot A\underline{x} \geq 0$

Chapter 5

Vector Valued Function

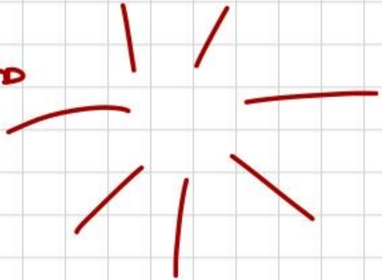
Vector field: $f : D \subseteq \mathbb{R}^q \longrightarrow \mathbb{R}^q$

$$\underline{P_0} \longrightarrow \underline{w}_{P_0}$$



Continuous function

Characterisation of the limit



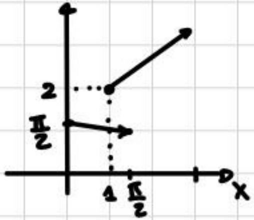
Continuity can be shifted into continuity of its component: No proof but

- how does it work?
- $\underline{x}_n \to \underline{x}$ convergent sequence in $\mathbb{R}^p$

Exercise 1

$$F(x,y) = \langle x^2+y , e^x \sin y \rangle$$

- $F(1,2) = \langle 3, e\sin 2 \rangle$
- $F(0, \frac{\pi}{2}) = (\frac{\pi}{2}, 1)$

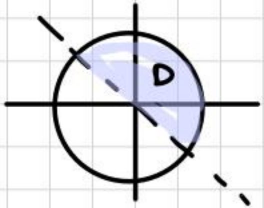


Exercise 2

Find the domain of

$$F(x,y) = \langle \ln(x+y), \sqrt{4-x^2-y^2} \rangle$$

$$\begin{cases} x+y > 0 \\ 4-x^2-y^2 \geq 0 \end{cases} \quad \begin{cases} y > -x \\ x^2+y^2 \leq 4 \end{cases}$$



Differential Calculus

PARTIAL DERIVATIVES

Exercise 5

$\frac{\partial F}{\partial x'}$, $\frac{\partial F}{\partial y}$ for $F(x,y) = (x^2y + e^y, \sin(xy))$

$\frac{\partial F}{\partial x} = (2xy , y\cos(xy))$ instead $\nabla F_1 (2xy , x^2 + e^y)$

$\frac{\partial F}{\partial y} = (x^2 + e^y , x\cos(xy))$ $\nabla F_2 (y\cos(xy), x\cos(xy))$

A CONVENTION:

If f is a column function · $f : D \subseteq \mathbb{R}^p \to \mathbb{R}^q$

$$\begin{pmatrix} \underline{v} \\ \\ \end{pmatrix} \longmapsto \begin{pmatrix} \underline{w} \\ \\ \end{pmatrix} = \begin{pmatrix} f(\underline{v}) \\ \\ \end{pmatrix} = \begin{pmatrix} f_1(\underline{v}) \\ f_2(\underline{v}) \\ \vdots \end{pmatrix} \qquad \nabla f_1 \boxed{\qquad\qquad\qquad}$$

n° col. ⇒ dim (p)
n° rows. ⇒ dim (q)

Exercise 4

$$W = \begin{pmatrix} x^2 + 3xy + 6z^2 \\ xy + 2\ln\frac{z}{x} \end{pmatrix}$$

- $\frac{\partial W}{\partial x} = \begin{pmatrix} 2x+3y \\ y - \frac{2z}{x^2} \cdot \frac{x}{z} \end{pmatrix} = \begin{pmatrix} 2x+3y \\ y - \frac{2}{x} \end{pmatrix}$

- $\frac{\partial W}{\partial y} = \begin{pmatrix} 3x \\ x \end{pmatrix}$

- $\frac{\partial W}{\partial z} = \begin{pmatrix} 12z \\ \frac{2}{x} \cdot \frac{x}{z} \end{pmatrix} = \begin{pmatrix} 12z \\ \frac{2}{z} \end{pmatrix}$

$\nabla W_1 = (2x + 3y,\ 3x,\ 12z)$

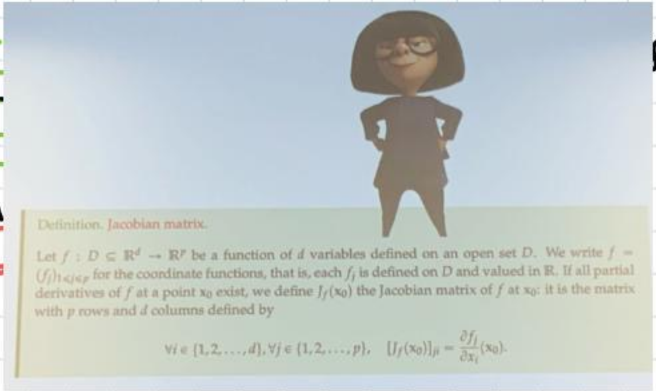
$\nabla W_2 = (y - \frac{2}{x},\ x,\ \frac{2}{z})$

The Jacobian Matrix

$f: D \subseteq \mathbb{R}^d \to \mathbb{R}^p$

$$J_f(x_0) = \begin{bmatrix} -\nabla f_1- \\ -\nabla f_2- \\ \vdots \\ -\nabla f_p- \end{bmatrix}$$

d co



Definition. Jacobian matrix.

Let $f : D \subseteq \mathbb{R}^d \to \mathbb{R}^p$ be a function of d variables defined on an open set D. We write $f = (f_j)_{1 \leq j \leq p}$ for the coordinate functions, that is, each f_j is defined on D and valued in $\mathbb{R}$. If all partial derivatives of f at a point $\mathbf{x}_0$ exist, we define $J_f(\mathbf{x}_0)$ the Jacobian matrix of f at $\mathbf{x}_0$: it is the matrix with p rows and d columns defined by

$$\forall i \in \{1, 2, \ldots, d\}, \forall j \in \{1, 2, \ldots, p\}, \quad [J_f(\mathbf{x}_0)]_{ji} = \frac{\partial f_j}{\partial x_i}(\mathbf{x}_0).$$

ferential of the vector valued function

Exercise 5

$$f(x,y,z) = \begin{pmatrix} \sqrt{z(x^2+y^2)} \\ \ln \frac{xy}{z} \end{pmatrix} \qquad D = \begin{cases} z > 0 \\ x \cdot y > 0 \end{cases}$$

writing the Jacobian Matrix

$D = \{(x,y,z) \in \mathbb{R}^3 \quad x \cdot y > 0 \ , \ z > 0\}$ open and unbounded

$$Jf(x,y,z) = \begin{bmatrix} \sqrt{z}\,\frac{2x}{2\sqrt{x^2+y^2}} \ ; & \\ & \ddots \ \text{Continua tu} \end{bmatrix}$$

Ex. 6

$$f(x,y) = \begin{pmatrix} \sin x \cos y \\ \sin x \sin y \\ \cos x \end{pmatrix} \qquad D(\text{potentially}) \text{ is } \mathbb{R}^2$$

function in several variables
that we used to describe the surface of the sphere

$$\begin{pmatrix} R\sin\varphi\cos\theta \\ R\sin\varphi\sin\theta \\ R\cos\varphi \end{pmatrix}$$

$$J_f(x,y) = \begin{bmatrix} \cos x \cos y & -\sin x \sin y \\ \cos x \sin y & \sin x \cos y \\ -\sin x & 0 \end{bmatrix}$$

Divergence of a vector valued function

the divergence of a field
$F=(F_1,F_2,F_3)$ in 3-D space is defined

$$\nabla\cdot F=\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$$

$$\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{pmatrix}\cdot\begin{pmatrix}F_1\\ F_2\\ F_3\end{pmatrix}=\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\quad : \quad \overset{\mathbb{R}^3}{\begin{pmatrix}x\\ y\\ z\end{pmatrix}}\longrightarrow \overset{\mathbb{R}}{\nabla\cdot F}$$

Curl of a vector valued function → (Only in $\mathbb{R}^3$)

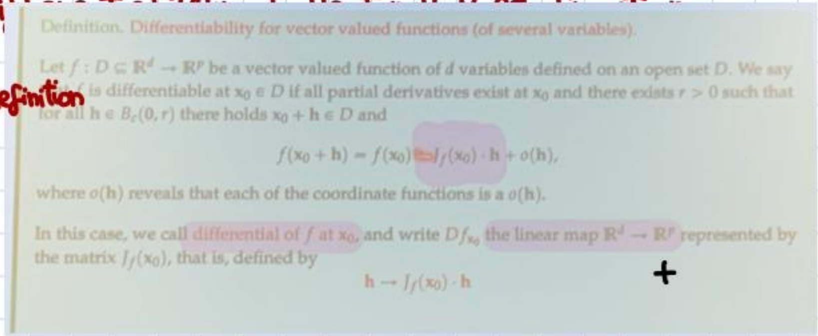
$$\nabla\times F=\det\begin{pmatrix} i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{pmatrix}=\underline{i}\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)-\underline{j}\left(\frac{\partial F_3}{\partial x}-\frac{\partial F_1}{\partial z}\right)+\underline{k}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)$$

$\underbrace{\text{whenever we talk of the curl in } \mathbb{R}^2 \text{ is a field}}_{\text{Plane vector field}}$ $F(x,y,0)$

$$\hookrightarrow \det\begin{pmatrix} i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_1(x,y) & F_2(x,y) & 0\end{pmatrix}=\underline{i}\cdot 0-j\ 0+\underline{k}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)$$

Di

1st definition



Definition. Differentiability for vector valued functions (of several variables).

Let $f : D \subseteq \mathbb{R}^d \to \mathbb{R}^p$ be a vector valued function of d variables defined on an open set D. We say f is differentiable at $\mathbf{x}_0 \in D$ if all partial derivatives exist at $\mathbf{x}_0$ and there exists $r > 0$ such that for all $\mathbf{h} \in B_\varepsilon(\mathbf{0}, r)$ there holds $\mathbf{x}_0 + \mathbf{h} \in D$ and

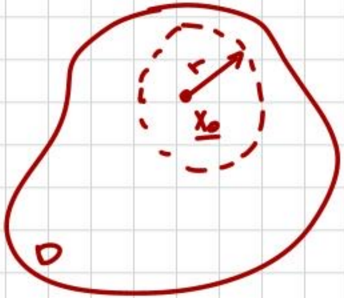
$$f(\mathbf{x}_0 + \mathbf{h}) = f(\mathbf{x}_0) + J_f(\mathbf{x}_0) \cdot \mathbf{h} + o(\mathbf{h}),$$

where $o(\mathbf{h})$ reveals that each of the coordinate functions is a $o(\mathbf{h})$.

In this case, we call differential of f at $\mathbf{x}_0$, and write $Df_{\mathbf{x}_0}$ the linear map $\mathbb{R}^d \to \mathbb{R}^p$ represented by the matrix $J_f(\mathbf{x}_0)$, that is, defined by

$$\mathbf{h} \to J_f(\mathbf{x}_0) \cdot \mathbf{h}$$

h = displacement

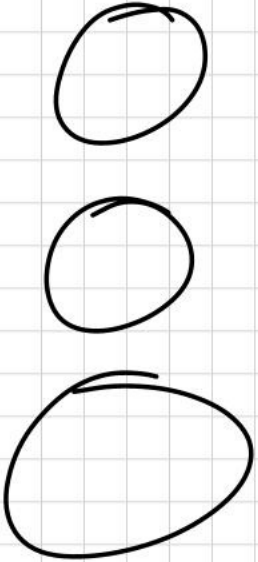


J_f is the representative Matrix of the Differential

2nd ...

Curve in Parametric form

Remember the connection between Curves and differentiable functions



Ex. 9

• $f : R^2 \to R^3$ • $g : R^3 \to R^2$

$$f(u,v) = \begin{pmatrix} u^2 \\ v \\ u+v \end{pmatrix} \qquad f(u,v,z) = \begin{pmatrix} x+y \\ yz \end{pmatrix}$$

we want to compute the differential of the composition $g\circ f : R^2 \to R^2$ at the point $(u,v) = (1,2)$

$$f(1,2) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad g(1,2,1+2) = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

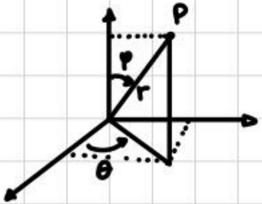
$$\begin{tikzcd} J_f(u,v) = \begin{pmatrix} 2u & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \arrow[d] & J_g(u,v) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & z & y \end{pmatrix} \arrow[d] \\ (1,2) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} & (1,2,3) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 2 \end{pmatrix} \end{tikzcd}$$

$$\Rightarrow J_{g\circ f}(1,2) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 5 \end{pmatrix}$$

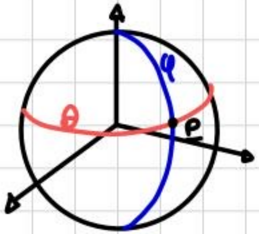
$$\bullet\ g\big(f(1,2)\big) = \begin{pmatrix} u^2 + v \\ uv + v^2 \end{pmatrix}$$

Exercise 19

- **Sphere** : Providing a description and a suitable parametrization



$$\begin{cases} x = (r\sin\varphi)\cos\theta \\ y = (r\sin\varphi)\sin\theta \\ z = r\cos\varphi \end{cases}$$



$$f : \overset{D}{[0,2\pi) \times [0,\pi]} \longrightarrow \mathbb{R}^3$$

$$(0,\varphi) \longmapsto f(\theta,\varphi) = (R\sin\varphi\cos\theta, R\sin\varphi\sin\theta, R\cos\varphi)$$

- Determine the direction that is orthogonal to both coordinate curves Namely the direction normal to the surface of the sphere.

(2 columns of the Jacobian Matrix)

$$\frac{\partial f}{\partial \theta}(\theta,\varphi) = (-R\sin\varphi\sin\theta, R\sin\varphi\cos\theta, 0)$$

$$\frac{\partial f}{\partial \varphi}(\theta,\varphi) = (R\cos\varphi\cos\theta, R\cos\varphi\sin\theta, -R\sin\varphi)$$

$$\underbrace{\frac{\partial f}{\partial \theta} \times \frac{\partial f}{\partial \varphi}}_{\text{Normal to the tangent plane}} = \begin{vmatrix} i & j & K \\ -R\sin\varphi\sin\theta & R\sin\varphi\cos\theta & 0 \\ R\cos\varphi\cos\theta & R\cos\varphi\sin\theta & -R\sin\varphi \end{vmatrix}$$

$$= \left(-R^2\sin^2\varphi\cos\theta \quad -R^2\sin^2\varphi\sin\theta \quad -R^2\sin\varphi\cos\varphi\sin^2\theta + R^2\sin\varphi\cos\varphi\cos^2\theta\right) =$$

$$= -R^2\sin\varphi\left(\sin\varphi\cos\theta - \sin\varphi\sin\theta - \cos\varphi\sin^2\theta + \cos\varphi\cos^2\theta\right)$$

Part 6: higher order derivatives

we define:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$$ second derivative

formal type:

$$\frac{\partial^2 f}{\partial x \partial y} \quad \text{or} \quad f_{yx}$$ "first y then x"

k-th derivative:

$$\frac{\partial^k f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}} = \frac{\partial}{\partial x_{i_1}}\left(\frac{\partial}{\partial x_{i_2}}\left(\cdots\left(\frac{\partial f}{\partial x_{i_k}}\right)\right)\right)$$

Exercise 1

d) $f(x,y,z) = xy + 2yz + 4y^2 + z^2$

$f_x = y$ $\quad f_y = x + 2z + 8y$ $\quad f_z = 2y + 2z$

$f_{xx} = 0$ $\quad \boxed{f_{xy} = 1}$ $\quad \boxed{f_{xz} = 0}$

$f_{yy} = 8$ $\quad \boxed{f_{yx} = 1}$ $\quad \boxed{f_{yz} = 2}$

$f_{zz} = 2$ $\quad \boxed{f_{zy} = 2}$ $\quad \boxed{f_{zx} = 0}$

Mixed derivatives coincide always?

Schwarz's Theorem

$$\frac{\partial^2 f}{\partial x \partial y}(x_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0) \quad \text{iff} \quad f \in C^2(\overset{\text{open}}{D})$$

- f is continuous of at every point of D
- f has 1st and 2nd order der. cont. at D
- all partial derivatives of f are diff.

Strong Concequence of Schwarz's Theorem: Hessian Matrix

Hessian Matrix : H_f

$$\begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}$$

from Exercise 1 we have $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 8 & 2 \\ 0 & 2 & 2 \end{bmatrix} \rightarrow 2 \cdot \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 4 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

still Ex 1.

a) $f_{xx} = e^{x+y^2}$

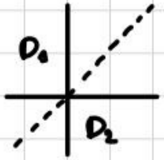
$f_{xy} = 2y\, e^{x+y^2}$

$f_{yy} = \left(2y\, e^{x+y^2}\right)' = 2e^{x+y^2} + 4y^2 e^{x+y^2}$ $\rightarrow H_f : \begin{bmatrix} e^{x+y^2} & 2y\, e^{x+y^2} \\ 2y\, e^{x+y^2} & 2e^{x+y^2}(1+2y^2) \end{bmatrix} = e^{x+y^2} \begin{bmatrix} 1 & 2y \\ 2y & 2+4y^2 \end{bmatrix}$

$f_{yx} = 2y e^{x+y^2}$

c) $f_3(x,y) = \ln \frac{x^2+y^2}{x-y} \longrightarrow f_3 \in C^2(\underset{\downarrow}{D})$

$(D_1 \cup D_2)$



About the gradient of a potential

$g=(g_1,g_2)$ $\exists$ some $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ scalar function $g=\nabla f$ if so f is the potential of g

Idea use Schwarz's theorem

$$\begin{array}{l} g_1=\frac{\partial f}{\partial x} \searrow \\ g_2=\frac{\partial f}{\partial y} \nearrow \end{array} \boxed{\frac{\partial}{\partial y}\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\frac{\partial f}{\partial y}}$$

condition on cross derivatives

necessary condition for $g=\nabla f$ is that $\frac{\partial}{\partial y}g_1=\frac{\partial}{\partial x}g_2$ where $g_1=\frac{\partial f}{\partial x}$, $g_2=\frac{\partial f}{\partial y}$

Generalising the condition for a vector field in $\mathbb{R}^3$

$g=(g_1,g_2,g_3)$

$$\nabla\times g=\det\begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ g_1 & g_2 & g_3 \end{pmatrix}=\underline{i}\left(\frac{\partial g_3}{\partial y}\underset{\substack{\shortparallel\\0}}{-}\frac{\partial g_2}{\partial z}\right)-\underline{j}\left(\frac{\partial g_3}{\partial x}\underset{\substack{\shortparallel\\0}}{-}\frac{\partial g_1}{\partial z}\right)+\underline{k}\left(\frac{\partial g_2}{\partial x}\underset{\substack{\shortparallel\\0}}{-}\frac{\partial g_1}{\partial y}\right)$$

$g_1=\frac{\partial f}{\partial x}$

$g_2=\frac{\partial f}{\partial y}$

$g_3=\frac{\partial f}{\partial z}$

Schwarz's theorem general case

Ex. 3

$$f(x,y) = x^3y - 2xy + 5xy^4$$

$f_x = 3x^2y - 2y + 5y^4$

$f_{xx} = 6xy$

$f_{xy} = 3x^2 - 2 + 20y^3$

$f_{xxx} = 6y$

$f_{xxy} = 6x$

$f \quad = 120$

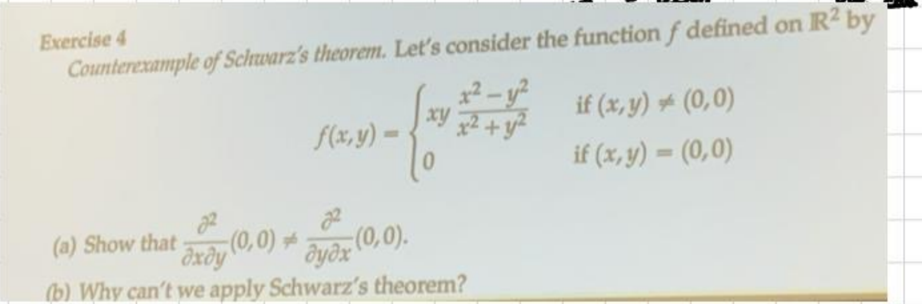
I know from the theorem:

- $f_{xxy} = f_{yxx} = f_{xyx} = 6x$
- $f_{yyx} = f_{xyy} = f_{yxy} = 60y^2$

We can organize them into a Tensor

↳ A cube of numbers

$$\boxed{T_{ijk} = \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}}$$



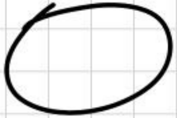
Exercise 4

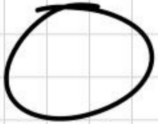
Counterexample of Schwarz's theorem. Let's consider the function f defined on $\mathbb{R}^2$ by

$$f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) Show that $\dfrac{\partial^2}{\partial x \partial y}(0,0) \neq \dfrac{\partial^2}{\partial y \partial x}(0,0)$.

(b) Why can't we apply Schwarz's theorem?





Second Order Taylor Expansion

approx of f at $\underline{P_0}$: $T_1(x,y) = f(P_0) + f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0)$

$$T_2(x,y) = T_1(x,y) + \frac{1}{2}\left[f_{xx}(\underline{P_0})(x-x_0)^2 + f_{xy}(P_0)(x-x_0)(y-y_0) + f_{yx}(P_0)(x-x_0)(y-y_0) + f_{yy}(\underline{P_0})(y-y_0)^2\right]$$

$$f(x,y) \overset{\downarrow}{\underset{\uparrow}{=}} T_2(x,y) + \underset{\|\underline{h}\|^2 + o(1)}{o(\underline{h}^2)_{\shortparallel}}$$

Chain Rule Again:

$f \in D \subseteq C^2$ open

Find the T_2 exp. at $P^* = (1,1)$

$f = xy^2 - 2xy$ $\Rightarrow T_1(1,1) = -1 - 1 + 0 = -2$

$f(P^*) = -1$ $\Rightarrow T_2(1,1) = -2 + \frac{1}{2}\left[2(x-1)(y-1) + 2(x-1)(y-1) + 2(y-1)^2\right]$

$f_x(P^*) = -1$

$f_y(P^*) = 0$

$f_{xx} = 0$

$f_{yy} = 2x \to f_{yy}(P^*) = 2$

$f_{xy} = f_{yx}$ because $f \in C^2$ $= 2y$

$f_{xy}(P^*) = 2$

quadratic form

$\downarrow$

$q(v) \to v = \begin{bmatrix} x-1 \\ y-1 \end{bmatrix}$

describes the movement close to the point

$v \cdot Av$ the writing of the quadratic form

$A = A^T$ where $A = H_f$

The Compact Vector Notation:

$$f(x_0 + h) = f(x_0) + \nabla f(x_0) \cdot h + \frac{1}{2} h \cdot H_f(x_0) \cdot \underline{h} + o(\|h\|^2)$$

$\hookrightarrow$ vector of the increments

Optimization Problem

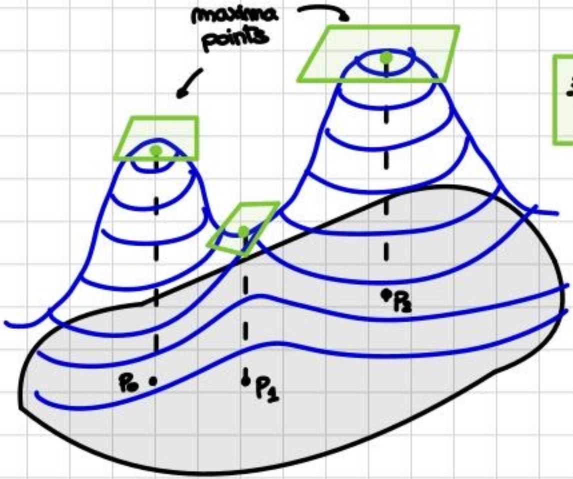
max $f(x,y)$

free $\underline{x} \in D_{(open)}$

Def of maximum (local): $\exists\, r>0$ s.t. $\underbrace{f(P_0)}_{\text{Max value}} \geq f(P) \quad \forall P \in B_c(P_0, r)$

Global maximum: $f(P_0) \geq f(P) \quad \forall \underline{P} \in D$

optimal points $\begin{cases} P_0 & \text{maximiser} \\ P_0 & \text{minimiser} \end{cases}$



$$\boxed{z = f|_{P_0}} + \cancelto{0}{f_x|_{\underline{P_0}}}(x-x_0) + \cancelto{0}{f_y|_{\underline{P_0}}}(y-y_0)$$

$$\nabla f(\underline{P_0}) = \underline{0}$$

Fermat Necessary and not sufficient condition for extrema

if $f'(P_0) = 0$

How to generalise the first order suff. cond
↓
sign of first derivative to study if max or min ⟶ It makes no sense in higher dim: $\mathbb{R}^2$ IS NOT ORDERED

The setting of our theory is "differentiable"
smoothness

f differentiable $\nabla f(P_0) = 0 \Rightarrow P_0$ stationary points
$\Updownarrow$ or critical points
$f \in C^1(D)$

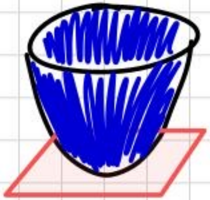
Fermat theorem candidate optimal

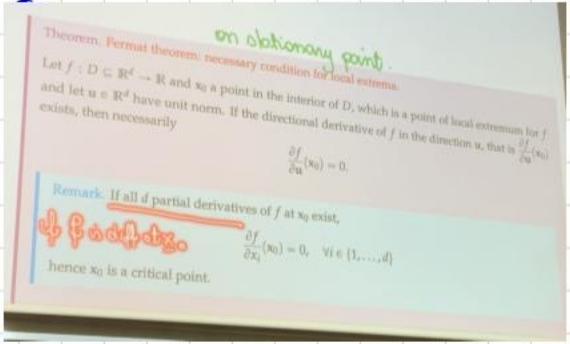
$$f(x,y) = \underbrace{T_1(x,y)}_{\text{costant}} + \sigma(P_0)$$

$\Rightarrow$ not sufficient I want an higher order approximation

$f \in C^2(D)$ $$f(x,y) = f(P_0) + f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) + \frac{1}{2}\left[\underbrace{f_{xx}(P_0)(x-x_0)^2 + 2f_{xy}(x-x_0)(y-y_0) + f_{yy}(y-y_0)^2}\right] + o(P_0^2)$$

is P is a local maximiser $f(x,y) \geq f(P_0)$? Study the sign of the quadratic form!





on stationary point.

Theorem. Fermat theorem: necessary condition for local extrema.

Let $f : D \subseteq \mathbb{R}^d \to \mathbb{R}$ and x_0 a point in the interior of D, which is a point of local extremum for f and let $u \in \mathbb{R}^d$ have unit norm. If the directional derivative of f in the direction u, that is $\frac{\partial f}{\partial u}(x_0)$ exists, then necessarily

$$\frac{\partial f}{\partial u}(x_0) = 0.$$

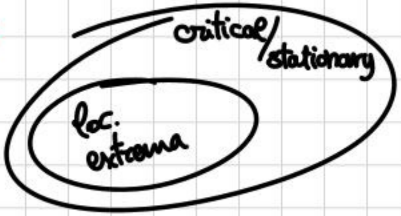
Remark. If all d partial derivatives of f at x_0 exist,

if f is diff at x_0

$$\frac{\partial f}{\partial x_i}(x_0) = 0, \quad \forall i \in \{1, \ldots, d\}$$

hence x_0 is a critical point.

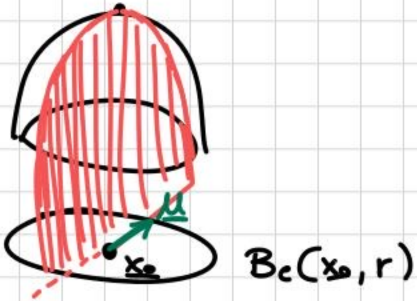
f is differentiable



PROOF: $d > 1$

$$t \longmapsto f(\gamma(t)) = g(t)$$

$$\gamma(t) = \underline{x_0} + t\,\underline{u}$$



it has an optimal point at $\underline{x_0}$
i.e. for $t = 0$.

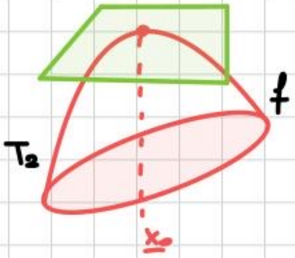
$$g'(0) = \lim_{t \to 0} \frac{g(0+t) - g(0)}{t} = \frac{\partial f}{\partial \underline{u}}(x_0)$$

□

How to study whether max or min:
study the sign of the Hessian Matrix at $\underline{x_0}$
meaning the sign of the quadratic form associated to $H_f(x_0)$

- Positive definite : strict loca minimum
- Negative definite : strict local maximum
- Indefinite : Saddle point

Proof: $f \in C^2(D)$ (Schwartz's theorem) $\Rightarrow Hf$ is symmetric + we can use $T_2(\underline{x}_0)$



① If $Hf(\underline{x}_0)$ is neg. def. $\Rightarrow \underline{x}_0$ is a local maximizer

$$f(x,y) = T_2(x,y) + o(\underbrace{(x-x_0)^2+(y-y_0)^2}) = \|\underline{h}\|^2$$

$$\hookrightarrow \underline{h} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}$$

$$= f(\underline{x}_0) + \cancel{\nabla f(\underline{x}_0)} \cdot (\underline{x}-\underline{x}_0) + \frac{1}{2}(\underline{x}-\underline{x}_0) \cdot Hf(\underline{x}_0)\,(\underline{x}-\underline{x}_0) + o(\|\underline{h}\|^2)$$

$\rightarrow 0$ because of the stationarity

$$\Rightarrow f(\underline{x}) - f(\underline{x}_0) = \frac{1}{2}(\underline{x}-\underline{x}_0) \cdot Hf(\underline{x}_0) \cdot (\underline{x}-\underline{x}_0) + o(\|\underline{h}\|^2)$$

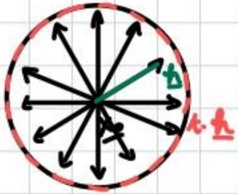
now I divide everything by $(\|h\|^2)$

$$\frac{f(x) - f(x_0)}{\|h\|^2} = \frac{1}{2}\,\frac{(\underline{x}-x_0) \cdot Hf(\underline{x}_0) \cdot (\underline{x}-\underline{x}_0)}{\|h\|^2} + o(1)$$

we use h as an increment and unitary vector and we pass to polar coordinates

$$= \frac{1}{2}\,\frac{(\cancel{t}\underline{h}) \cdot Hf(\underline{x}_0) \cdot (\cancel{t}\underline{h})}{\|\cancel{t}h\|^2} + o(1)$$

$\Rightarrow$ It doesn't depend on t but only on the values in $B_0(\underline{x}_0, h)$



$$= \underbrace{\frac{1}{2}q(\underline{h})} + o(1)$$

where q is the quadratic form associated to $H_f(\underline{x_0}) \Rightarrow \underline{h}\cdot H_f(\underline{x}_0)\cdot \underline{h}$

$\Rightarrow$ The sign of the slope near x_0 depends on the sign of $H_f(x_0)$

Level set at maximum

the maxima points will be on the SMALLEST hellips intercepting $q(\underline{h})$

the minima points will be on the LARGEST hellips intercepting $q(\underline{h})$

$$\Rightarrow \frac{f(\underline{x})-f(\underline{x}_0)}{\|h\|^2} = \frac{1}{2}q(h) + o(1) \leq \frac{M}{2} + \underset{\downarrow \; < \frac{|M|}{4}}{\boxed{o(1)}} \leq \frac{M}{4}$$

negative

$\lim_{t \to 0}$

$$\begin{array}{c} 0\,? \\ \vee \\ M \\ \vee\!| \\ q(\underline{h}) \\ \vee\!| \\ m \end{array}$$

Exercise 6

$$f(x,y) = -2x^2 - y^2 + 3(x+y) - xy + 3 \qquad f(x,y) \in C^2(\mathbb{R}^2)$$

solve the problem: $\max\limits_{(x,y)\in\mathbb{R}^2} f(x,y)$

$$\nabla f(x,y) = 0 \Rightarrow \begin{cases} f_x = -4x + 3 - y = 0 \ ; \ y = 3 - 4x \xrightarrow{\qquad\qquad\qquad} y = \frac{9}{7} \\ f_y = -2y + 3 - x = 0 \ ; \ -6 + 8x + 3 - x = 0, \ 7x - 3 = 0 \ ; \ x = \frac{3}{7} \end{cases} \Rightarrow \text{stationary point } P\left(\tfrac{3}{7}, \tfrac{9}{7}\right)$$

$$\downarrow$$

$$\begin{cases} 4x + y = 3 \\ x + 2y = 3 \end{cases} \qquad A = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\underline{P_0} = A^{-1} \cdot \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \frac{1}{\det(A)} \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 3 \\ 9 \end{pmatrix} = \begin{pmatrix} \frac{3}{7} \\ \frac{9}{7} \end{pmatrix}$$

Sign of the quadratic form depends on sign of the eigenvalues (spectral theorem)

$$e = \det(\lambda I - A) = 0 \Rightarrow \det \begin{pmatrix} \lambda - 4 & -1 \\ -1 & \lambda - 2 \end{pmatrix} = 0 \Rightarrow \begin{aligned} &(\lambda-4)(\lambda-2) - 1 = 0 \\ &\lambda^2 - 6\lambda - 7 = 0 \\ &\Delta = 8 \end{aligned} \quad \begin{cases} \frac{6 - \sqrt{8}}{2} \\ \frac{6 + \sqrt{8}}{2} \end{cases}$$

How to skip the computation of eigenvalues:

$\forall$ Matrix M , $\det(M) = \prod$ eigenvalues and the $tr(A) = \sum$ eigenvalues

In our case: $\det A > 0 \Rightarrow \lambda_1, \lambda_2$ are concordi

$tr(A) > 0 \Rightarrow \lambda_1 > 0$ and $\lambda_2 > 0$

$$\Downarrow$$

- $\det(H) > 0$ and $tr(H) > 0 \Rightarrow x_0$ strict local minimum
- $\det(H) > 0$ and $tr(H) < 0 \Rightarrow x_0$ strict local maximum
- $\det(H) < 0 \Rightarrow x_0$ saddle point

!!! $\Rightarrow$

$f''(x_0) > 0$ $\cup$

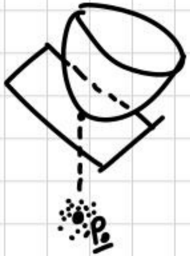
$f''(x_0) < 0$ $\cap$

$f''(x_0) = 0$??? this case is inconclusive

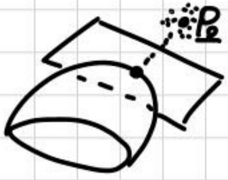
what holds at $\underline{x_0}$ holds everywhere
stationary

$$f(x,y) = f(\underline{P_0}) + \nabla f(P_0)\cdot(\underline{P}-\underline{P_0}) + \frac{1}{2}(\underline{P}-\underline{P_0})\cdot Hf(\underline{P_0})(\underline{P}-\underline{P_0}) + \sigma(h^2)$$

$$f(x,y) - \underbrace{\left(f(\underline{P_0}) + \nabla f(P_0)\cdot(\underline{P}-\underline{P_0})\right)}_{T_2} = \underbrace{\frac{1}{2}(\underline{P}-\underline{P_0})\cdot Hf(\underline{P_0})(\underline{P}-\underline{P_0})}_{\text{sign of it}} + \sigma(h^2)$$



if $f(\underline{P})$ is above or below tg plane at P_0
~~sufficient~~ condition for convexity of the function
(look at theorem of sign of H_f)



When it is not applicable: everytime $\det H_f(\underline{P_0}) = 0$

examples: surface with different local behaviour
but same (singular) hessian matrix ⟹ QUADRATIC FORMS

0 as eigenvalue

Let's see where it fails:

$C^\infty(\mathbb{R}^2)$
$f(x,y) = x^4 + y^4$
$g(x,y) = x^4 + y^3$
$T_2(x,y) = 0$ but one is as a minimum at $(0,0)$ and the other as a saddle point

/!\ My Alert is $\det H_f(\underline{P_0}) = 0$

Summary: for a 2,2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\det A = ad - bc = \lambda_1 \cdot \lambda_2$

$\text{tr } A = a + d = \lambda_1 + \lambda_2$

Exercise

$f(x,y) = x^2 - 2x + y^4 + y^2$ $\qquad f(x,y) \in C^2(R^2)$

$$\begin{cases} f_x = 2x - 2 = 0 \\ f_y = 4y^3 + 2y = 0 \end{cases} \Rightarrow \begin{array}{l} x = 1 \\ y(4y^2+2) = 0 \Rightarrow y = 0 \end{array} \Rightarrow \underline{P_0}(1,0)$$

$$Hf = \begin{pmatrix} 2 & 0 \\ 0 & 12y^2+2 \end{pmatrix} \qquad Hf(1,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$\lambda_1 = \lambda_2 = 2$

both positive

$\Rightarrow$ $\underline{P_0}$ is a local minimum

*

$\forall P \in R^2$, Hf is positive definite

$\Rightarrow$ P_0 global minimum

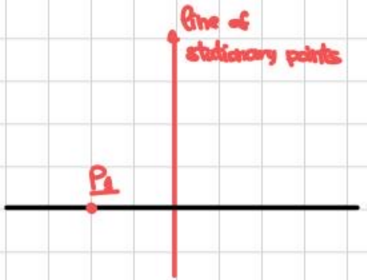
My surface is locally convex

and globally ?

*

Exercise

$$f(x,y) = x^2y(x-y+1) \qquad f(x,y) \in C^{\infty}(\mathbb{R}^2)$$



$$\begin{array}{l} f_x \\ f_y \end{array} \begin{cases} 3x^2y - 2xy^2 + 2xy = 0 ① \\ x^3 - 2yx^2 + x^2 = 0 \quad ② \end{cases} \begin{cases} xy(3x-2y+2) = 0 \\ x^2(x-2y+1) = 0 \end{cases} \Rightarrow \begin{array}{l} P_1(0;y) \\ P_2(-1,0) \end{array}$$

$$① \cdot \frac{x}{y} = ②$$

- if $x \neq 0,\ y = 0$ ① ✓
  ② : $x = -1$

P_3 :

- if $x \neq 0,\ y \neq 0$: $A = \begin{bmatrix} 3 & -2 \\ 1 & -2 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{-4} \begin{pmatrix} -2 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} +2 \\ +1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{4} \end{pmatrix}$

$$f_{xx} = 6xy - 2y^2 + 2y$$

$$f_{yy} = -2x^2$$

$$f_{yx} = 3x^2 - 4xy + 2x$$

$$Hf = \begin{pmatrix} 6xy - 2y^2 + 2y & 3x^2 - 4xy + 2x \\ 3x^2 - 4xy + 2x & -2x^2 \end{pmatrix}$$

$$Hf(0,y) = \begin{pmatrix} 2y(-y+1) & 0 \\ 0 & 0 \end{pmatrix}$$

$$Hf(-1,0) = \begin{pmatrix} 0 & 5 \\ 5 & -2 \end{pmatrix} \qquad Hf\left(-\frac{1}{2}, \frac{1}{4}\right) = \begin{pmatrix} -\frac{3}{8} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{2} \end{pmatrix}$$

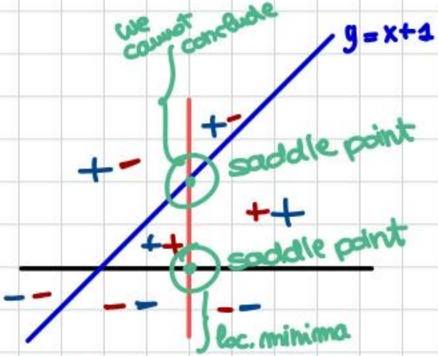
$\det Hf(P_1) = \lambda_1 \cdot \lambda_2 < 0 \Rightarrow$ maximum

$\det Hf(P_2) = \lambda_1 \cdot \lambda_2 > 0 \Rightarrow$ minimum

$\det Hf(P_3) = \lambda_1 \cdot \lambda_2 > 0 \Rightarrow$ maximum

How to study this line of st. points?

$$f(P) \substack{\le \\ \ge} \underset{\overset{\|}{0}}{f(\underline{P_1})}$$



$P_1(0,y)$ loc. maxima
$y > 1$

$P_1(0,y)$ loc. minima
$y < 0$

solving the inequality:

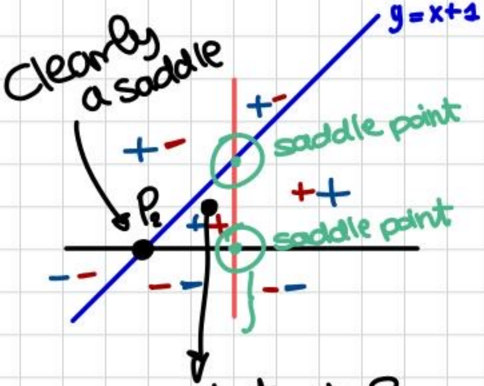
$$f(x,y) \ge 0$$

$$x^2 \underline{y} \, \underline{(x - y + 1)} \ge 0$$

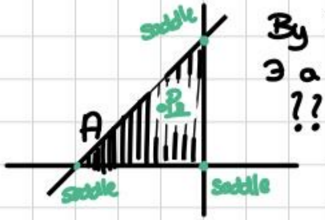
$$x^2 y \ge y - x - 1$$

$$x^2 y \ge y - x - 1$$

Can we conclude the nature of the other 2 stationary points in an elementary way as I did for P_1?



What abt P_3 : I use this domain A to restrict my function



By Weierstrass
$\exists$ a max and min
???

Exercise 9

$$f(x,y) = x^3-6x^2-y^2-4y+7$$

$$\begin{cases} f_x = 3x^2-12x = 0 \\ f_y = -2y-4 = 0 \end{cases} \qquad \begin{cases} x(3x-12) = 0 \begin{array}{l} 0 \\ 4 \end{array} \\ y = -2 \end{cases} \rightarrow \begin{array}{l} \underline{P_0} = (0,-2) \\ \underline{P_1} = (4,-2) \end{array}$$

$$f_{xx} = 6x-12$$

$$f_{yx} = 0$$

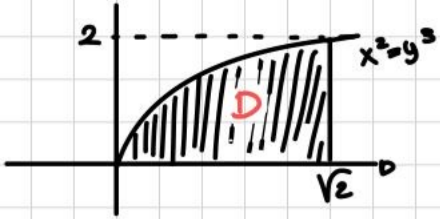
$$f_{yy} = -2$$

$$H_f \begin{pmatrix} 6x-12 & 0 \\ 0 & -2 \end{pmatrix} \rightarrow H_f(\underline{P_0}) = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \quad \det = 0$$

$$H_f(\underline{P_1}) = \begin{pmatrix} 12 & 0 \\ 0 & -2 \end{pmatrix} \quad \det < 0$$

Exercise 10

$f(x,y) = x^2 + y^2 - 2y + 1$



$f(x,y) \in D$ not free

↓

compact

closed and bounded

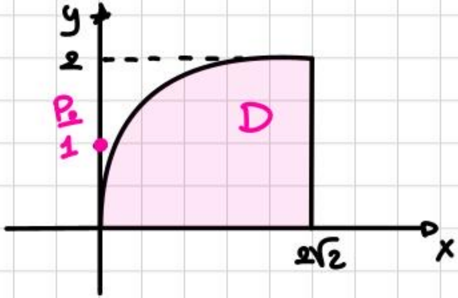
↳closed $\exists \partial D$

1) optimize f on $\dot{D}$

2) optimize f on ∂D → they make a closed curve γ

Split the problem into 2:

① on $\dot{D}$ the problem becomes a free optimization



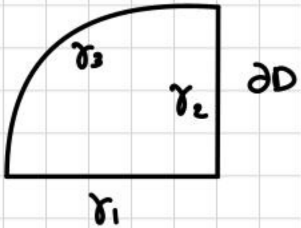
$$\begin{matrix} f_x \\ f_y \end{matrix} \begin{cases} 2x = 0 & \rightarrow x = 0 \\ 2y - 2 = 0 & \rightarrow y = 1 \end{cases}$$

$\underline{P_0}\ (0,1) \notin D$

we discard P_0 as candidate

But notice $\underset{\substack{\downarrow \\ \text{compact}}}{f \in C^0(D)}$ $\Rightarrow$ By Weiestrass we know $\exists$ $\begin{matrix} M & \text{maximisers} \\ m & \text{minimizers} \end{matrix}$

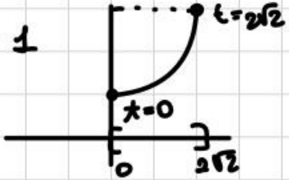
② Studying the boundary as a curve $\partial D = \gamma(t)$



$\partial D = \gamma_1 \cup \gamma_2 \cup \gamma_3 = \gamma$ continuous $f = x^2 + y^2 - 2y + 1$

- $\gamma_1 : t \to (t, 0)$ $t \in I = [0, 2\sqrt{2}]$

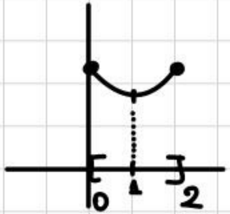
$f(\gamma_1(t)) = t^2 + 1$



$\to$ $\overset{?}{(2\sqrt{2}, 0)}$

- $\gamma_2 : t \to (2\sqrt{2}, t)$ $t \in [0, 2]$ $\to$ domain over the constraint of y

$f(\gamma_2(t)) = 8 + t^2 - 2t + 1 = 8 + (t+1)^2$



1 candidate for minimizer

2 candidate for maximizers

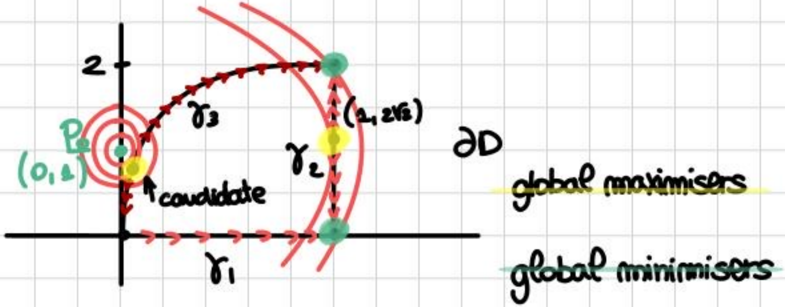
$(2\sqrt{2}, 1)$ $(2\sqrt{2}, 0)$ $(2\sqrt{2}, 2)$?

- $f\Big|^{\text{restrict}}_{x^2=y^3} = y^3 + y^2 - 2y + 1 \qquad 0 \le y \le 2$

$$\frac{d}{dy}\left(f\Big|_{x^2=y^3}\right) = 3y^2 + 2y - 2 \quad \text{among interior points} \quad 0 < y < 2$$

$$y_{1,2} = \frac{-1 \pm \sqrt{1+6}}{3}$$

$$y_1 = -\frac{1+\sqrt{7}}{3}$$



Path Integrals

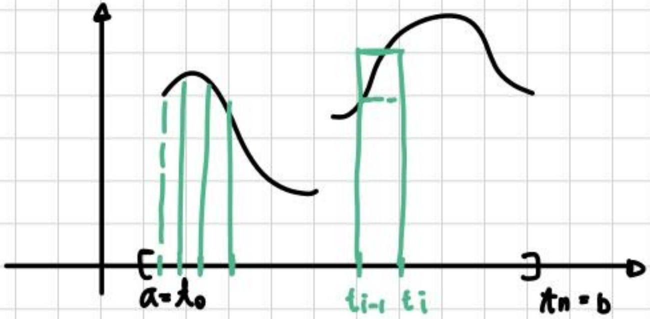
Recap of Riemman Integrals

Riemman Integrability = cumulating utility over $I = [a,b]$
(NOT. finding primitive)

1) Partition in n subintervals

inf. f.
$t \in [t_{i-1}, t_i]$

sup f.



$$\sum_{i=1}^{n} (\inf f)\cdot(t_i - t_{i-1}) \le \text{cumulative output} \le \sum_{i=1}^{n} (\sup f.)(t_i - t_{i-1})$$

lower Riemmann sum

upper sum

A function is integrable

$\int_a^b f(t)\,dt$ = inf. of upper R-sum
$|||$
sup. of lower R-sum

f is 1) R-integrable in $[a,b]$

2) has an anti-derivative on $[a,b]$

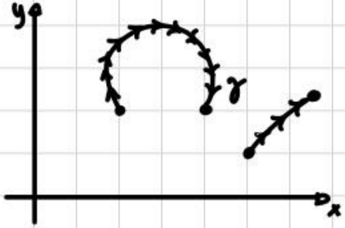
$[G \in C^1((a,b))$ s.t. $G'(t) = f(t)]$

then $\int_a^b f(t)\,dt = G(b) - G(a)$

Path Integrals:

Let

- $\gamma : I = [a,b] \to \mathbb{R}^d$ a curve of class C^1 and
- $f : \mathbb{R}^d \to \mathbb{R}$ a continuous (scalar) function

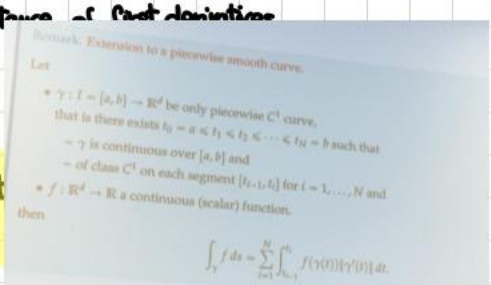


how much regularity we assume?

Continuity and continuity and existence of first derivatives

We Define Path Integrals

$$\int_\gamma f \, ds = \int_a^b f(\gamma(t)) \|\gamma'(t)\| \, dt$$



Remark. Extension to a piecewise smooth curve.

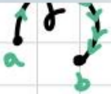
Let

- $\gamma : I = [a,b] \to \mathbb{R}^d$ be only piecewise C^1 curve, that is there exists $t_0 = a \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_N = b$ such that
  - γ is continuous over $[a,b]$ and
  - of class C^1 on each segment $[t_{i-1}, t_i]$ for $i = 1, \ldots, N$ and
- $f : \mathbb{R}^d \to \mathbb{R}$ a continuous (scalar) function.

then

$$\int_\gamma f \, ds = \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} f(\gamma(t)) \|\gamma'(t)\| \, dt.$$

↓ space travelled



If NOT of clas C^1 I can divide piecewise in C^1.

Exercise 1

$f(x,y) = x^2 + y^2$ and the straight path $(0,0) \to (1,2)$

$$\int_\gamma f \, ds$$

$\gamma : t \in [0,1] \longrightarrow (t, 2t)$

$f(x,y) \in C^1(D)$

$\gamma' = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad |\gamma'| = \sqrt{5}$

$$= \int_\gamma f(\gamma(t)) \|\gamma'(t)\| \, dt$$

$$= \int_0^1 \left[(t)^2 + (2t)^2 \right] \sqrt{5} \, dt = \sqrt{5} \cdot 5 \left[\frac{t^3}{3} + \frac{t^3}{3} \right]_0^1 = \sqrt{5} \cdot 5 \left(\frac{1}{2} + \frac{4}{3} \right) = \sqrt{5} \, \frac{10}{3}$$

Exercise 2

$f(x,y,z) = \frac{xz}{y}$ and $\gamma : t \in [1,2] \to \left(t, \sqrt{\frac{3}{2}} t^2, t^3 \right)$

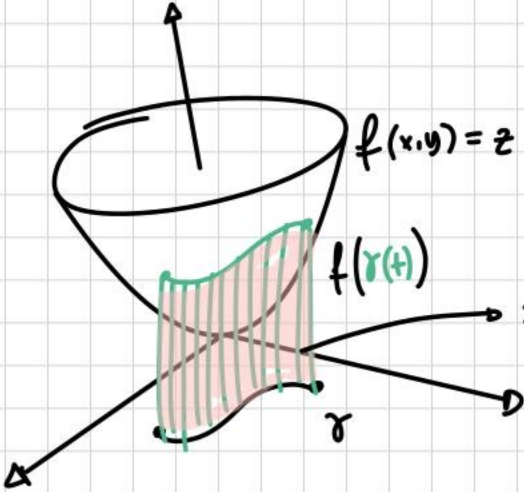
$$\int_\gamma f \, ds$$

$t \in [1,2] \quad \gamma(t) = \left(t, \sqrt{\frac{3}{2}} t^2, t^3 \right)$

$\gamma'(t) = \left(1, 2t\sqrt{\frac{3}{2}}, 3t^2 \right) \to \|\gamma'\| = \sqrt{1 + \overset{2}{\cancel{4}} t^2 \frac{3}{\cancel{2}} + 9t^4} = \sqrt{1 + 6t^2 + 9t^4}$

$$\int_1^2 \frac{t \cdot t^{\cancel{2}}}{\sqrt{\frac{3}{2}}\,\cancel{t^2}} \cdot \sqrt{\underbrace{1+6t^2+9t^4}_{\overset{\parallel}{(1+3t^2)^2}}}\,dt = \sqrt{\frac{2}{3}}\int_1^2 t^2(1+3t^2)\,dt = \sqrt{\frac{2}{3}}\left[\frac{t^3}{3}+\frac{3}{5}t^5\right]_1^2$$

Graphycally



I calculate this area

Connection with Riemman Integral

$$\int_\gamma f\, ds = \lim_{N\to+\infty} \sum_{k=0}^{N-1} f(\gamma(t_k^N)) \underbrace{\left\|\gamma(t_{k+1}^N)-\gamma(t_k^N)\right\|}$$

Applications:

Length of a Curve: $\int_\gamma 1\, ds$

$$\gamma: \begin{cases} x = R\cos t \\ y = R\sin t \end{cases} \qquad t \in [0, 2\pi]$$

$$\text{length}(\gamma) = \int_\gamma 1 \cdot ds$$

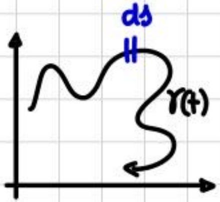
$$\gamma': \begin{cases} x = -R\sin t \\ y = R\cos t \end{cases}$$

$$\|\gamma'\| = R$$

Path Integral

Real Valued Function

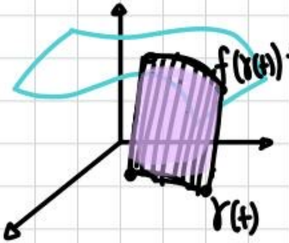
$$\iint_\gamma f(x,y)\, ds$$



Differential ⟶ scalar

$$\int_\gamma f(x,y)\, ds \overset{\text{Riemann Integral}}{=} \int_I f(\gamma(t))\cdot \overbrace{\|\gamma'\|\, dt}^{\text{Infinitesimal space travelled}}$$

Vector Valued function



Differential ⟶ Vector

I want to parametrize the same curve in different ways:

Example circumference $\gamma \begin{cases} x = 3\cos t \\ y = 3\sin t \end{cases}$ I need also I (domain of variability)

$$I = [0, 2\pi]$$

Another way: $\omega \begin{cases} x = 3\cos 2\tau \\ y = 3\sin 2\tau \end{cases} \rightarrow J = [0, \pi]$

$\gamma)$ $t \in [0, 2\pi] \longrightarrow \gamma(t) = (\dots)$

$\gamma(t) = \omega(s(t))$

$\varphi(t)$ connection transforming the parameter $\longrightarrow s(t)$

$\tau = s(t)$

$\tau = \frac{t}{2}$

a certain level of differentiability

Diffeomorphism : I, J two intervals of $\mathbb{R}$ and $k \geq 1$ is an int, we call diffeomorphism of class C^k

- a function $\varphi : I \to J$ of class C^k
- which is bijective
- and such that the inverse function $\varphi^{-1} : J \to I$ is also of class C^k.

Characterisation of Diffeomorphism:

$$\boxed{\varphi \text{ is a diffeomorphism } \iff \varphi' : I \to \mathbb{R} \text{ does not vanish}}$$

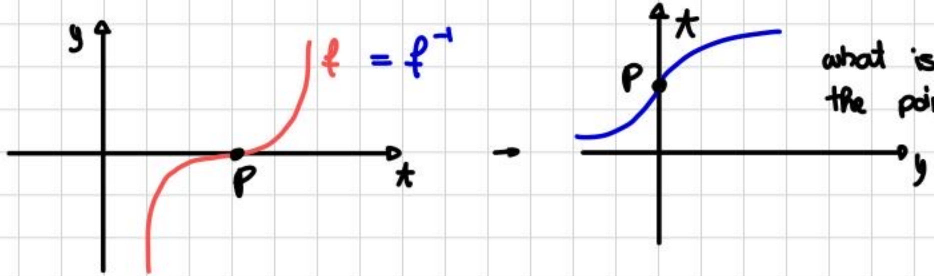
Increasing

$\varphi' > 0 \quad \forall t \in I$

Decreasing

$\varphi' < 0 \quad \forall t \in I$

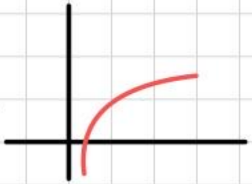
Proof: See book □



what is the problem: the point P

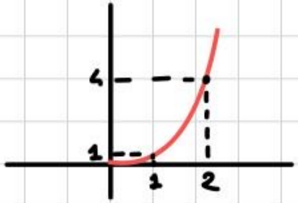
Increasing Diffeomorphism

- $\varphi: t \in \mathbb{R} \to 2t \in \mathbb{R}$
- $\psi: t \in (0, +\infty) \to \ln t \in \mathbb{R}$

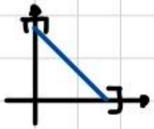


- $\xi: t \in [1,2] \to t^2 \in [1,4]$

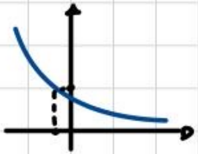
Parabola not invertible
but I can use just a portion



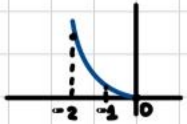
Decreasing Diffeomorphism



- $\eta: t \in [0,1] \to 2t \in \mathbb{R}$



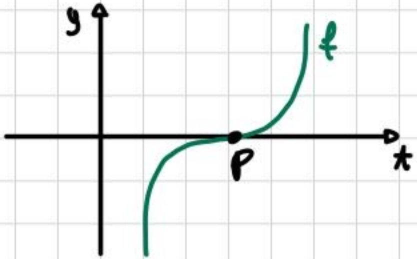
- $\lambda: t \in \to e^{-t} \in (0, +\infty)$
- $x: t \in [-2,-1] \to t^2 \in [1,4]$

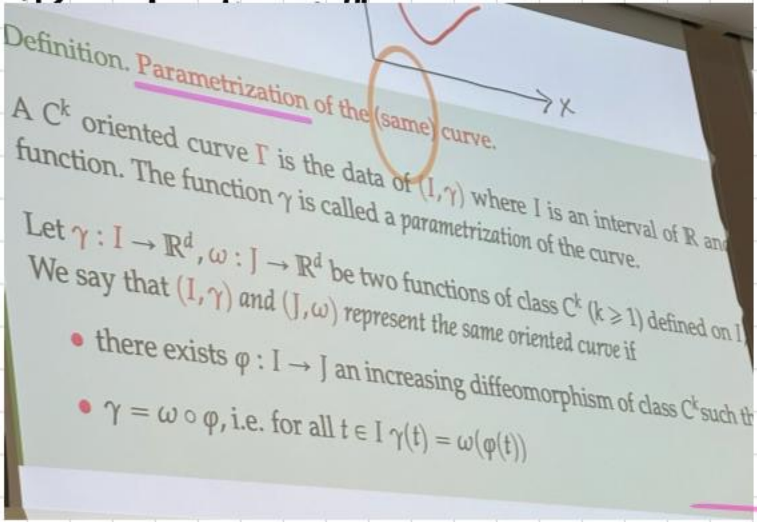


Bijective function that are not diffeomorphism

$\xi: t \in \mathbb{R} \to t^3 \in \mathbb{R}$

Indeed $\xi'(0) = 0$ and the inverse function $\xi: t \to t^{-\frac{1}{3}}$ is not differentiable at $t = 0$.



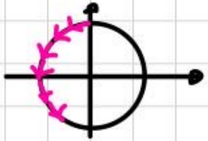


Definition. Parametrization of the (same) curve.

A C^k oriented curve Γ is the data of (I,γ) where I is an interval of $\mathbb{R}$ an
function. The function γ is called a parametrization of the curve.

Let $\gamma : I \to \mathbb{R}^d$, $\omega : J \to \mathbb{R}^d$ be two functions of class C^k $(k \geqslant 1)$ defined on I
We say that (I,γ) and (J,ω) *represent the same oriented curve if*

- there exists $\varphi : I \to J$ an increasing diffeomorphism of class C^ksuch th
- $\gamma = \omega \circ \varphi$, i.e. for all $t \in I$ $\gamma(t) = \omega(\varphi(t))$



"A circle travelled in a specific sense"
Clockwise ecc.

If they have opposite orientation $\Rightarrow$ $\exists$ a decreasing Diffeomorphism

Path Integral is Independent on the parametrization

2 functions $$\left.\begin{array}{l}\gamma: I \to \mathbb{R}^d\\ \omega: J \to \mathbb{R}^d\end{array}\right\rangle C^1$$ they represent the same oriented curve

Then, for any continuous function $f: \mathbb{R}^d \to \mathbb{R}$ we have that

$$\int_\gamma f\,ds = \int_\omega f\,ds$$

Proof: $$\int_\gamma f\,ds \overset{\text{df}}{=} \int_a^b f(\gamma(t))\|\gamma'(t)\|\,dt$$; Let φ a diffeomorphism $\varphi' > 0$ $\Rightarrow \gamma = \omega \circ \varphi$

$I = [a,b]$

$$= \int_a^b f(\omega(\varphi(t)))\cdot\|\omega'(\varphi(t))\|\,\overbrace{\varphi'(t)\,dt}^{d\tau}$$

substituting $\tau = \varphi(t)$ $\Rightarrow$ $$\int_{c=\varphi(a)}^{d=\varphi(b)} f(\omega(\tau))\cdot\|\omega'(\tau)\|\,d\tau$$

$\Vert$

$$= \int_\omega f\,ds$$

$$\gamma'(t) = \frac{d}{dt}\left(\omega(\varphi(t))\right) = \omega'(\varphi(t))\cdot\varphi'(t)$$

$$\Rightarrow \|\gamma'(t)\| = \|\omega'(\varphi(t))\cdot\varphi'(t)\| = \varphi'(t)\|\omega'(\varphi(t))\|$$

Now: γ and ω represent Opposite oriented curves

$\Rightarrow$ the Diffeomorphism is decreasing

what changes? when substituting $\tau = \varphi(t) \Rightarrow$

$$-\int_{d=\varphi(b)}^{c=\varphi(a)} f(\omega(\tau)) \cdot \tau' \cdot \|\omega'(\tau)\| \, d\tau = -\int_{d}^{c} f(\omega(\tau)) \cdot \|\omega'(\tau)\| \, d\tau$$

$$= \int_{\omega} f \, ds$$

Nothing changes!

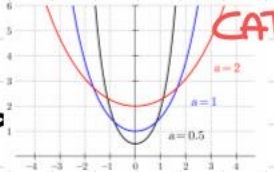
The Theorem holds in its integrity!

$\downarrow$ Why?

Length of a curve $\Rightarrow$ $\gamma : I \to \mathbb{R}$ $\gamma \in C^1$ its lenght is

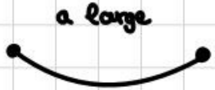
$$\int_{\gamma} ds = \int_{I} \|\gamma'(t)\| \, dt$$

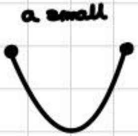
Exercise 4



CATENARY

y = a co





Length of y : [-1, 1]

Step 1) <u>Parametrize</u> : every function in chartesian form can be parametrized.

$$\gamma : t \in I = [-1,1] \longrightarrow \gamma(t) = (t, \cosh t)$$

$$\gamma'(t) = (1, \sinh t)$$

$$\|\gamma'(t)\| = \sqrt{1+\sinh^2 t}$$

⇒ Lenght: $\displaystyle\int_{-1}^{1}\sqrt{1+\sinh^2 t}\,dt = \int_{-1}^{1}\sqrt{\cosh^2 t}\,dt = 2\int_{0}^{1}\cosh t\,dt$

$$= 2\Big[\sinh t\Big]_0^1 = 2\sinh(1)$$

I know:

$$\cosh^2 - \sinh^2 = 1$$

$$\left(\frac{e^t+e^{-t}}{2}\right)^2 - \left(\frac{e^t-e^{-t}}{2}\right)^2 = 1$$

A SPECIAL PARAMETRIZATION: THE NORMAL PARAMETRIZATION

Let Γ be an oriented curve and let (I, γ) be one of its parametrization
the parametrization is said to be normal if $\|\gamma'(t)\| = 1$.

$\gamma'(t) \neq \underline{0} \ \forall t$

If your curve is regular ∃ a Normal Parametrization (if the curve is smooth I can always decide to travel it at costant speed.)

$$\gamma : t \in \underset{I}{\underset{\|}{[a,b]}} \longmapsto \gamma(t)$$

- path covered up to time t. $s(t) = ($lenght - path covered from $t=a, t)$. (lenght of a curve)

$$s(t) = \int_a^t \|\gamma'(t)\| \, dt$$

By I^{st} fundamental th. of Int. Calculus:
$s(t)$ continuous and $s(t) \in C^1$; $s'(t) = \|\gamma'(t)\| > 0$

- Now, from $s(t)$ deduce $t = t(s)$
(its inverse)

$s(t)$ is invertible

normal parametrization:

$$\omega(s) = \gamma(t(s))$$

Exercise 5

$$\gamma'(t) = (\cancel{\cos t} - t\sin t - \cancel{\cos t}, \cancel{\sin t} + t\cos t - \cancel{\sin t})$$

$$\|\gamma'(t)\| = \sqrt{t^2 \sin t^2 + t^2 \cos t^2} = \sqrt{t^2} = |t| = t \in [0, 2\pi]$$

$$s(t) = \int_0^t \tau \, d\tau = \frac{t^2}{2} \xrightarrow{\text{the inverse}} t = \sqrt{2s}$$

$$\Big\downarrow \text{normal par.}$$

$$\omega(s) = \gamma(t(s)) = \left(\sqrt{2s}\cos\sqrt{2s} - \sin\sqrt{2s}, \sqrt{2s}\sin\sqrt{2s} + \cos\sqrt{2s}\right)$$

why is this the normal parametrization

Theorem: EXISTANCE OF NORMAL PARAMETRIZATION

Let Γ be an oriented curve and let (I, γ) be one of its parametrization. We assume that all points of γ are regular, that is, $\gamma'(t) \neq 0 \ \forall t \in I$. Then $\exists \ \omega: J \to \mathbb{R}^d$ a function of class C^k s.t. (J, ω) is a normal parametrization of Γ.

Exercise 6

$I = [0, +\infty)$

$$\gamma(t) = \left(t^2, 2t, \frac{4\sqrt{2}}{3}\sqrt{t^3}\right)$$

$$\gamma'(t) = \left(2t, 2, \frac{\overset{2}{\cancel{4}}\sqrt{2}}{\cancel{3}}\frac{\cancel{3}}{\cancel{2}}\sqrt{t}\right)$$

$$\overset{s'(t)}{\overset{\parallel}{\|\gamma'(t)\|}} = \sqrt{4t^2 + 4 + 8t} = 2\sqrt{t^2 + 2t + 1} = 2\sqrt{(t+1)^2} = 2|t+1|$$

$$\Rightarrow s(t) = \int_0^t 2(\tau+1)\,dt = 2\left(\frac{t^2}{2} + t\right) = t^2 + 2t$$

now: $t^2 + 2t - s = 0$

$$t = \frac{-1 + \sqrt{1-s}}{1}$$ only sol. with $\oplus$

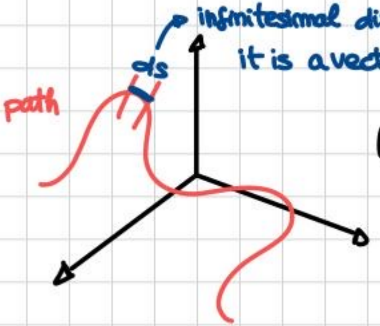
normal parametrization: $\omega(s) = \gamma\left(\frac{-1+\sqrt{1-s}}{1}\right)$

Integral of a Vector Field

f vector field

endomorphism $f : R^3 \longrightarrow R^3$

$\underline{x} \longrightarrow \underline{f}$ at $\underline{x}$



Work of f as a force

$dW = f \bullet d\vec{s}$

⇒ Overall Work along γ

$$\int_\gamma f \bullet \underline{ds} = \int_I f(\gamma(t)) \bullet \gamma'(t)\, dt = \sum_i^d \int_I f_i(\gamma(t))\, \gamma_i'(t)\, dt$$

$$\downarrow$$

$$\int_I f_1(\gamma(t)) \overbrace{\gamma_1'(t)\, dt}^{dx} + \int_I f_2(\gamma(t)) \overbrace{\gamma_2'(t)\, dt}^{dy} + \int_I f_3(\gamma(t)) \overbrace{\gamma_3'(t)\, dt}^{dz}$$

for $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$

$ds = (\gamma_1'(t), \gamma_2'(t), \gamma_3'(t))$

Exercise 7

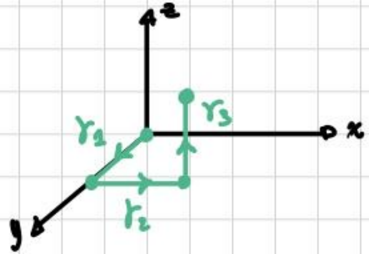
$$F = (xy, yz, x+y)$$

$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ which connects $(0,0,0) \longrightarrow (1,1,1)$

$$\gamma_1 \begin{cases} x = t \\ y = 0 \\ z = 0 \end{cases} \quad t \in I = [0,1] \qquad dx = dt$$

$$\gamma_2 \begin{cases} x = 1 \\ y = q \\ z = 0 \end{cases} \quad q \in I' = [0,1] \qquad \begin{matrix} dx = dz = 0 \\ dy = dq \end{matrix}$$

$$\gamma_3 \begin{cases} x = 1 \\ y = 1 \\ z = \tau \end{cases} \quad \tau \in [0,1] \qquad \begin{matrix} dx = dy = 0 \\ dz = d\tau \end{matrix}$$

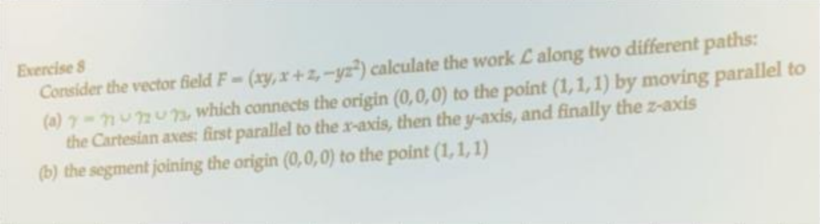


$$\int_\gamma F \cdot d\underline{s} = \int_{\gamma_1} \underline{F} \cdot d\underline{s} + \int_{\gamma_2} \underline{F} \cdot d\underline{s} + \int_{\gamma_3} \underline{F} \cdot d\underline{s}$$

$$\updownarrow$$

$$= \int_{\gamma_1} F_1\, dx + \underbrace{\int_{\gamma_2} F_2\, dy}_{\overset{\shortparallel}{0}} + \int_{\gamma_3} F_3\, dz$$

$$= \underbrace{\int F_1(\gamma_1) \cdot \gamma'(t)\, dt}_{\overset{\shortparallel}{0}} + \int_0^1 2\, dt = 2$$



Exercise 8

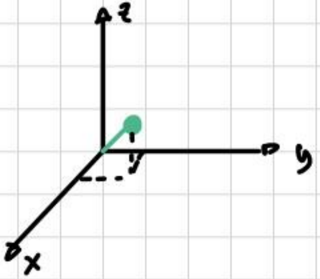
Consider the vector field $F = (xy, x+z, -yz^2)$ calculate the work $\mathcal{L}$ along two different paths:

(a) $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$, which connects the origin $(0,0,0)$ to the point $(1,1,1)$ by moving parallel to the Cartesian axes: first parallel to the x-axis, then the y-axis, and finally the z-axis

(b) the segment joining the origin $(0,0,0)$ to the point $(1,1,1)$

WORK OF A FORCE DEPENDS ON THE PATH

b)



$(0,0,0) \to (1,1,1)$

$$\gamma = (\underset{x}{\overset{\shortparallel}{t}}, \underset{y}{\overset{\shortparallel}{t}}, \underset{z}{\overset{\shortparallel}{t}})$$

$dx = dy = dz = dt$

$$\int_\gamma F \bullet ds = \int_I F_1(\gamma_1) \cdot \gamma_1' \, dt + \int_I F_2(\gamma_2) \gamma_2' \, dt + \int_I F_3(\gamma_3) \gamma_3' \, dt$$

$$= \int_0^1 t^2 \, dt + \int_0^1 2t + \int_0^1 -t^3 \, dt = \frac{1}{3} + 1 - \frac{1}{4} = \frac{13}{12}$$

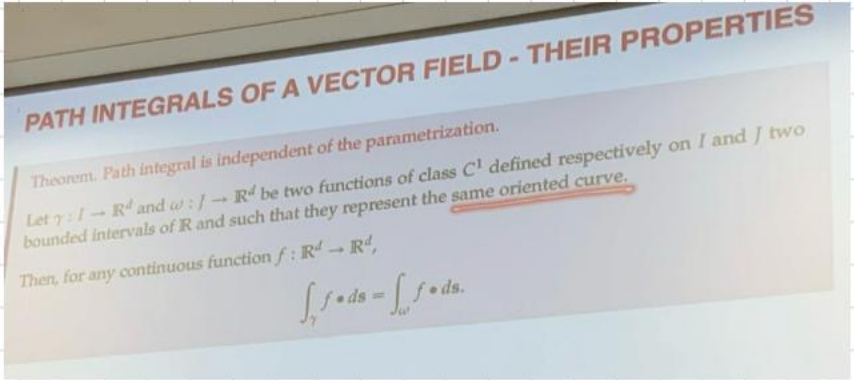
Exercise 9

$\gamma : t \in [-1,2] \to (t, t^2)$

$f(x,y) = (xy, x+y)$ then calculate $\int_\gamma F \cdot ds$

$$\|$$

$$\int_{-1}^{2} t^3 \cdot 1\, dt + \int_{-1}^{2} (t^2+t)\, 2t\, dt =$$



PATH INTEGRALS OF A VECTOR FIELD - THEIR PROPERTIES

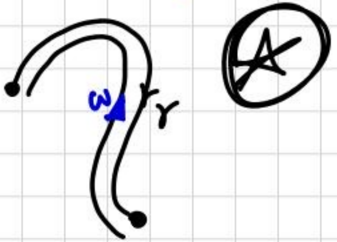
Theorem. Path integral is independent of the parametrization.

Let $\gamma : I \to \mathbb{R}^d$ and $\omega : J \to \mathbb{R}^d$ be two functions of class C^1 defined respectively on I and J two bounded intervals of $\mathbb{R}$ and such that they represent the same oriented curve.

Then, for any continuous function $f : \mathbb{R}^d \to \mathbb{R}^d$,

$$\int_\gamma f \bullet ds = \int_\omega f \bullet ds.$$

Remark : if



$$\int_\gamma f\cdot ds = \int_I f(\gamma(t))\cdot\gamma'(t)\,dt = \sum_i^d \int_I f_i(\gamma(t))\,\gamma_i'(t)\,dt$$

since γ and ω param. of the same curve ∃ a diffeomorphism

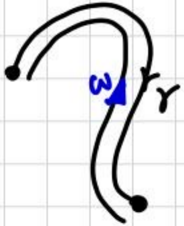
$$= \sum_i^d \int_J f_i(\omega(t))\,\omega_i'(t)\,dt = \int_J f(\omega(t))\cdot\omega'(t)\,dt$$

$$= \int_\omega f\cdot ds$$

Maybe Wrong check the proof

✪ If ω and γ have opposite orientation there is still dependency

$$\int_\gamma f\cdot\underline{ds} = \int_I f(\gamma)\cdot\gamma'(t)\,dt$$

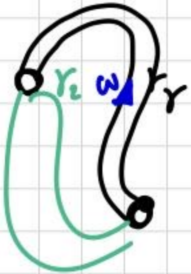


Decreasing φ diffeomorphism ?

$$\int_I f(\gamma(t))\cdot\gamma'(t)\,dt = \int f(\omega(\underbrace{\varphi(t)}_{\tau}))\cdot\omega'(\underbrace{\varphi(t)}_{\tau})\cdot\underbrace{\varphi'(t)\,dt}_{d\tau}$$

$$= \int_{d=\varphi(a)}^{c=\varphi(b)} f(\omega(\tau))\cdot\omega'(\tau)\,d\tau$$

2nd REMARK



Work is dependent on the path
UNLESS

↓

$$f = \nabla g$$ the gradient function (g is the potential)

In this case the calculation is path independent

$$\boxed{\int_\gamma \nabla g\cdot ds = g(\gamma(b)) - g(\gamma(a)) = [g\circ\gamma]_a^b}$$

difference of the potential

Proof: $\int_\gamma \nabla g \cdot ds = \int_I \underbrace{\nabla g(\gamma(t))\cdot \gamma'(t)}_{g'(t)}\, dt = g(\gamma(b)) - g(\gamma(a))$

By I fundamental theorem of int. calculus

$$g\circ\gamma(t) = g(\gamma(t))$$

$$\frac{d}{dt}\quad g : t \longrightarrow g(\gamma(t))$$

chain rule: $g'(t) = \nabla g(\gamma(t)) \cdot \gamma'(t)$

Theorem: Path Integral of a gradient field over a closed curve

$$\int_\gamma \nabla g \cdot ds = 0$$

EXAMPLE: how to "find" a potential
searching for a potential where the domain is $R^3 - D$

$$f = (f_1, f_2, f_3) = \nabla g \quad \text{IF} \quad \overbrace{\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x} \qquad \frac{\partial f_2}{\partial z} = \frac{\partial f_3}{\partial y} \qquad \frac{\partial f_3}{\partial x} = \frac{\partial f_1}{\partial z}}^{\text{Necessary condition}}$$

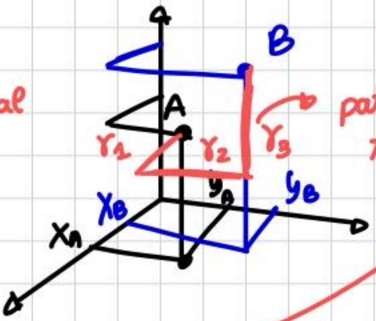
Is there a formula for the potential?

$$\int \nabla g \cdot \underline{ds} = g(\gamma(b)) - g(\gamma(a)) \quad \forall\ \gamma \left(\text{connecting } A=\gamma(a) \text{ to } B=\gamma(b)\right)$$

(Path INdependent)

I choose point $A = 0$

I will write a formula for the potential where I more parallel to the axis



path parallel to the axes

$dx = dt$

$$\int_\gamma \underline{f} \cdot \underline{ds} = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3}$$

$$= \int_{x_A}^{x_B} f_1(t, y_A, z_A) \cdot dt + \int_{y_A}^{y_B} f_2(x_B, t, z_A)\, dt + \int_{z_A}^{z_B} f_3(x_B, y_B, t)\, dt$$

γ_1: x varias from x_A to x_B

$\underset{y_A}{y}$ and $\underset{z_A}{z}$ are costant

γ_2: y varies from $y_A \to y_B$

$x = x_B$; $z = z_A$

γ_3: z varias from $z_A \to z_B$

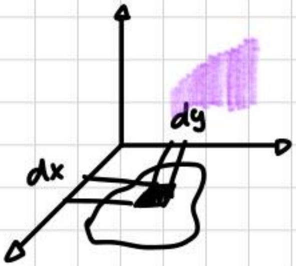
$x = x_B$, $y = y_B$

I let free the point B :

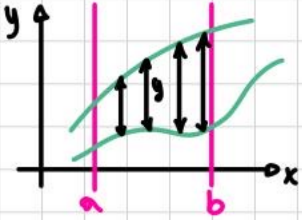
$$= \int_{x_A}^{x} f_1(t, y_A, z_A) \cdot dt + \int_{y_A}^{y} f_2(x, t, z_A)\, dt + \int_{z_A}^{z} f_3(x, y, t)\, dt$$

→ formula for the potential

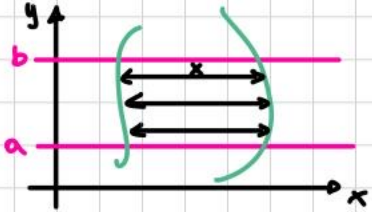
Integration of Several Variables



• Type 1



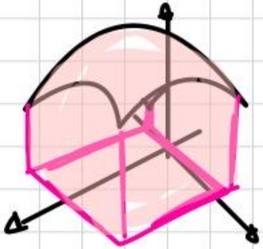
• Type 2



INTUITION : $f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$

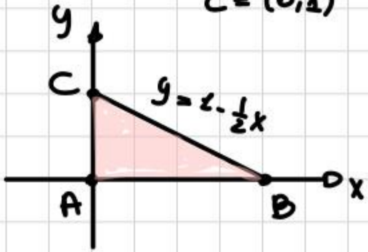
if $f > 0$ $\iint_D f(x,y)\,dx\,dy$

volume under the graph f projecting on D



Exercise 1

$A=(0,0)$
$B=(2,0)$
$C=(0,1)$



$y=1-\frac{1}{2}x$

$x=2-2y$

Type I:

$$\iint_D xy \, dx dy = \int_A^B x\left(\int_0^{1-\frac{1}{2}x} y\,dy\right)dx = \int_0^2\left(x\frac{1}{2}-\frac{x^2}{2}+\frac{x^3}{4}\right)dx = 1-\frac{8}{6}+16$$

$$\left[\frac{y^2}{2}\right]_0^{1-\frac{1}{2}x} \qquad \frac{\left(1-\frac{1}{2}x\right)^2}{2}=\frac{1}{2}-\frac{x}{2}+\frac{x^2}{4}$$

Type II:

$$\iint_D xy \, dx dy = \int_0^1 y\left(\int_0^{2-2y} x\,dx\right)dy =$$

Second Technique – Change of Variables

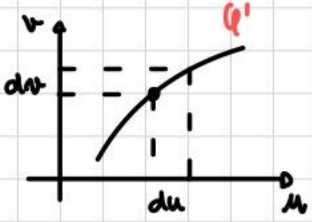
Check Notes

$$\iint_D f(x,y)\,dxdy = \iint_{\mathcal{U}} f(\varphi(u,v))\,|\det D\varphi(u,v)|\,du\,dv$$

will involve φ diffeomorphism

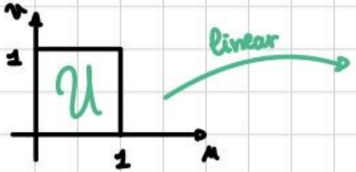
$$\begin{cases} x = \varphi_1(u,v) \\ y = \varphi_2(u,v) \end{cases}$$

The idea is using Diff. of φ → Represented by the Jacobian Matrix



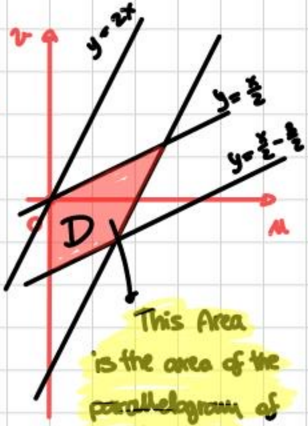
$$\begin{pmatrix} \frac{\partial \varphi_1}{\partial u} & \frac{\partial \varphi_1}{\partial v} \\ \frac{\partial \varphi_2}{\partial u} & \frac{\partial \varphi_2}{\partial v} \end{pmatrix}$$

TRANSFORMATIONS:



$$\begin{cases} x = 2u - v \\ y = u - 2v \end{cases}$$

$$J = \begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix}$$



This Area is the area of the parallelogram of the |det J|

(Voglio ottenere $^1\square_1$ nella trasformazione)

transformation of

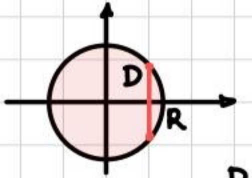
- u-axis: $v = 0 \longrightarrow \begin{cases} x = 2u \\ y = u \end{cases} \rightarrow x = 2y \;,\; y = \frac{x}{2}$
- $v = 1 \longrightarrow \begin{cases} x = 2u - 1 \\ y = u - 2 \end{cases} \Rightarrow y = \frac{x}{2} - \frac{3}{2}$
- $u = 0 \longrightarrow \begin{cases} x = -v \\ y = -2v \end{cases} \Rightarrow y = 2x$
- $u = 1 \longrightarrow \begin{cases} x = 2 - v \\ y = 1 - 2v \end{cases} \Rightarrow y = 1 - 2(2 - x)$

$y = 2x - 3$

Exercise 2

Use a substitution to calculate the area of the circle of radius R.

$$\underbrace{dx \cdot dy}_{dA} = |\det J| \, du \cdot dv$$



$$\iint_D 1 = ?$$

$$D = \{-R \le x \le R : -\sqrt{R^2-x^2} \le y \le \sqrt{R^2-x^2}\}$$

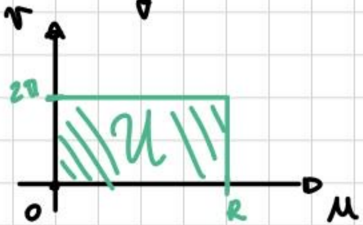
↓

Hard to be written as a domain of Type 1 or type 2

⇒ Using polar coordinates:

$$\begin{cases} x = u\cos v \\ y = u\sin v \end{cases} \qquad \begin{array}{l} u \in [0,R] \\ v \in [0,2\pi] \end{array}$$

↓



$$J = \begin{pmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{pmatrix}$$

$$|\det J| = u$$

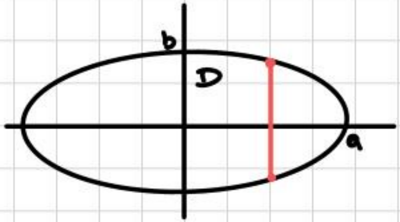
$$\Rightarrow \iint_D 1 = \iint_U u \, du \cdot dv$$

$$= \int_0^R u \, du \cdot \int_0^{2\pi} dv = \frac{R^2}{2} \cdot 2\pi = R^2\pi$$

For Polar Coordinates: $dx\,dy = \rho \, d\rho \, d\theta$

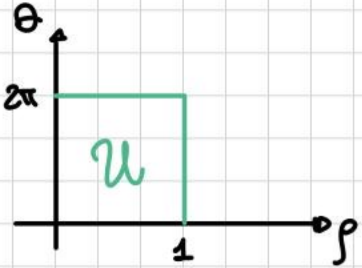
Exercise 3

Substitution for the area



$\iint_D 1$

$$\begin{cases} x = a\rho\cos\theta \\ y = b\rho\sin\theta \end{cases}$$



↳ up to 1 rescaled to "a" in x and "b" in y

$$J = \begin{pmatrix} a\cos\theta & -a\rho\sin\theta \\ b\sin\theta & b\rho\cos\theta \end{pmatrix} \rightarrow |\det J| = ab\rho\cos^2\theta + ab\rho\sin^2\theta$$

$$= ab\rho$$

$$\Downarrow$$

$$\iint_D 1 = \iint_U ab\rho \, d\rho d\theta$$

$$= \int_0^1 ab\rho \, d\rho \int_0^{2\pi} d\theta = ab\frac{1}{2}2\pi = ab\pi$$

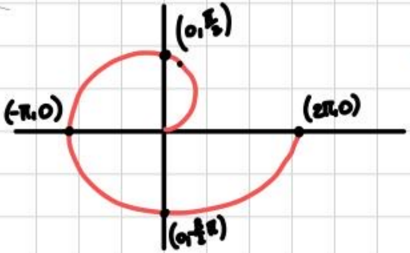
Exercise

bound of the curve $r=\theta \quad \theta\in[0,2\pi]$

$$\downarrow$$

$$\begin{cases} x=\theta\cos\theta \\ y=\theta\sin\theta \end{cases}$$

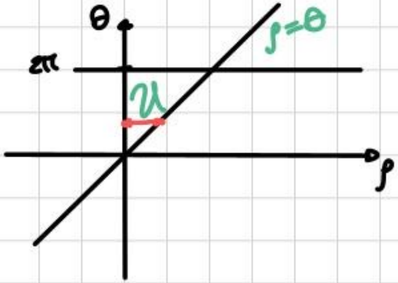
this curve is the boundary



Notice : this region is not of type I nor type II

$\Rightarrow$ Pass directly to polar coordinates.

the shape of $\mathcal{U}$:

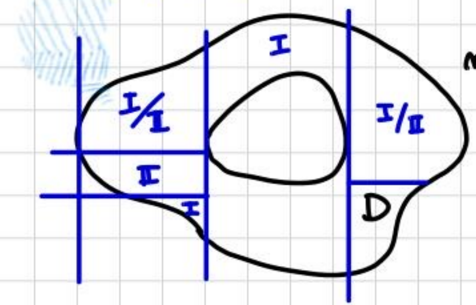


$0\leq\theta\leq2\pi$

$0\leq\rho\leq\theta$

$$\text{area}(D)=\iint_D 1\,dD=\iint_{\mathcal{U}}\rho\,d\rho\,d\theta=\int_0^{2\pi}\left(\int_0^{\theta}\rho\,d\rho\right)d\theta=\int_0^{2\pi}\frac{\theta^2}{2}d\theta=\frac{1}{6}8\pi^3=\frac{4}{3}\pi^3$$

Extending Fubini's theorem



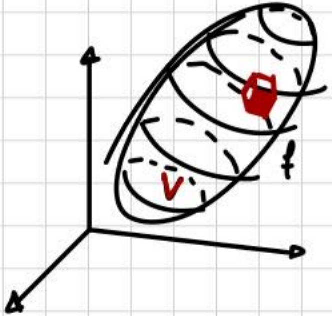
many types of Domain

For 3 variables

FUBINI

→ I want to find a Shape s.t. I can reduce the integration

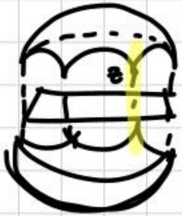
$$\underset{D=\text{Volume}}{\iiint} f(x,y,z)\ \underset{\substack{\downarrow\\ dx\cdot dy\cdot dz}}{dV}$$



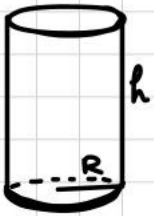
$\Sigma f(x_i,y_i,z_i)\cdot\Delta V$

→ Shape

- 1 + 2 (first in 2 and then in 1) → Type I "Slicing" → $\displaystyle\int_a^b \overbrace{\left(\iint_{D_z} f(x,y,z)\,dx\,dy\right)}^{\text{THE SLICE}} dz$
- 2+1 (first in 1 and then in 2) → Type II "Integration along a fiber" → $\displaystyle\iint_D \left(\int_{\alpha(x,y)}^{\beta(x,y)} f(x,y,z)\,dz\right) dx\,dy$



Volume of a Cylinder :



$$\iiint_V 1 =$$

Slicing : $\int_0^h \left(\iint_{D_z} 1 \, dx dy \right) dz = area(D) \cdot h$

here the slice won't depend on z

Exercise 5

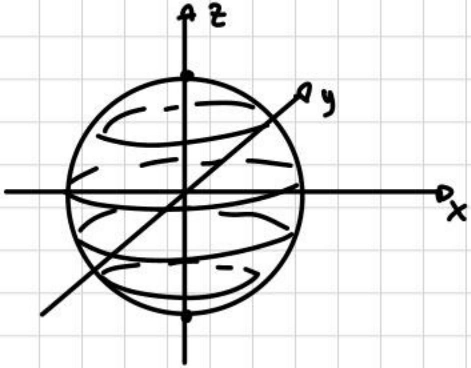
Compute the volume of the sphere

b) 2 + 1

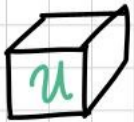
$$\int_0^{2R} \left(\iint_{D_z} 1 dxdy \right) dz \; ; \; \pi \int_0^{2R} z^2 dz = \frac{2}{3} \pi R^3$$

$\downarrow$

$D_z \Rightarrow x^2 + y^2 = z^2$



c) Change of coordinates : Spheric Coordinates



$0 \leq \rho \leq R$

$0 \leq \theta \leq 2\pi$

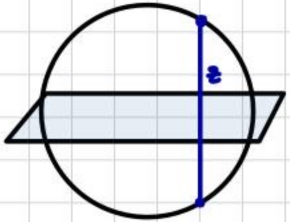
$0 \leq \varphi \leq \pi$

$$\iiint_V 1 \, dV = \iiint_U ? \; d\rho \, d\theta$$

$J = ($

ESERCIZIO:
Volume Sfera in 3 modi

a) Along a fiber: $x^2+y^2+z^2=R^2$



$$\alpha(x,y) = -\sqrt{R^2-x^2-y^2}$$

$$\beta(x,y) = +\sqrt{R+x^2+y^2}$$

$$\text{Volume}(V) = \iiint_V 1\,dV = \iint_D \left(\int_\alpha^\beta 1\cdot dz\right)$$

$D \hookrightarrow x^2+y^2=R^2$

$$= \iint_D \left(2\sqrt{R^2-x^2-y^2}\right) dx\,dy$$

change use polar

Det|J|

$$= \iint_U 2\sqrt{R^2-\rho^2}\,\boxed{\rho}\,d\rho\,d\theta$$

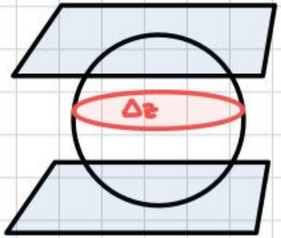
$U=(0,R)\times(0,2\pi)$

$$= 2\cdot 2\pi\int_0^R \sqrt{R^2-\rho^2}\,\rho\,d\rho = 4\pi\left[\frac{1}{3}(R^2-\rho^2)^{\frac{3}{2}}\right]_0^R$$

$$= \frac{4}{3}\pi R^3$$

---

b) Slicing Method:



$$\iiint_V 1\,dV = \int_{-R}^{+R}\left(\iint_{D_z} 1\,dx\,dy\right)dz$$

D_z: $\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le R^2-z^2\}$

$$= \int_{-R}^{+R}\pi(R^2-z^2)\,dz = 2\pi R^3 - \frac{2R^3}{3}\pi$$

$$= \frac{4}{3}\pi R^3$$

---

c) Spheric Coordinates:

$$\begin{cases} x = \rho\sin\varphi\cos\theta & f(\rho,\varphi,\theta) \\ y = \rho\sin\varphi\cos\varphi & g(\rho,\varphi,\theta) \\ z = \rho\cos\varphi & h(\rho,\varphi,\theta) \end{cases}$$

$$dV = dx\cdot dy\cdot dz = |\det J|\,d\rho\,d\theta\,d\varphi$$

$$\begin{bmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\varphi & 0 & -\rho\sin\varphi \end{bmatrix} = \det(J) = +\cos\varphi\left(-\rho^2\sin\varphi\cos\varphi\sin^2\theta - \rho^2\sin\varphi\cos\varphi\cos^2\theta\right)$$

$$+0 - \rho\sin\varphi\left(\rho\sin^2\varphi\cos^2\theta + \rho\sin^2\varphi\sin^2\theta\right)$$

$$dV = \left|-\rho^2 \sin\varphi\right| \, d\rho \, d\theta \, d\varphi$$

$$= -\cos\varphi\left(\rho^2 \sin\varphi\cos\varphi\right) - \rho\sin\varphi\left(\rho\sin^2\varphi\right)$$

$$dV = \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= -\left(\rho^2 \sin\varphi\cos^2\varphi + \rho^2 \sin\varphi \sin^2\varphi\right)$$

$$\boxed{= -\rho^2 \sin\varphi}$$

$$\text{Volume}(V) = \iiint_V 1 \, dV = \iiint_{U:\varphi\in[0,\pi],\theta\in[0,2\pi],\rho\in[0,R]} \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi = \int_0^{\pi} \sin\varphi \, d\varphi \int_0^{2\pi} d\theta \int_0^{R} \rho^2 d\rho$$

$$\downarrow$$

$$2 \cdot 2\pi \cdot \frac{R^3}{3} = \frac{4}{3}\pi R^3$$

Exercise 6

$$F(x,y,z) = x \cdot r i + y \cdot r j + z \cdot r k \qquad \text{where} \quad r = r(x,y,z) = \sqrt{x^2+y^2+z^2}$$

a) Verify it can be seen as a gradient field:

- Cross derivatives Identity

$$(\nabla \times f) = \underline{0} \qquad \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \; ; \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} \; ; \qquad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$$

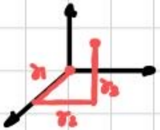
$$r = \sqrt{x^2+y^2+z^2}$$

$$F_1 = xr \qquad\qquad F_2 = y \cdot r$$

$$\frac{\partial F_1}{\partial y} = x \cdot r_y = \boxed{x \cdot \frac{y}{\sqrt{x^2+y^2+z^2}}} \qquad \frac{\partial F_2}{\partial x} = y \cdot r_x = \boxed{y \cdot \frac{x}{\sqrt{x^2+y^2+z^2}}} \; \checkmark$$

b) In case, find a potential g for F. Find some $g(x,y,z)$ s.t. $\nabla g = \underline{F}$ using path independence of $\int_\gamma \underline{F}\cdot \underline{ds} = \int_\gamma \nabla g \cdot \underline{ds} = g(\gamma(b)) - g(\underset{\overset{\|}{0}}{\gamma(a)})$

Pick $A = \gamma(a) = \underline{0}$



$$g(B) = \int_{\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3} \underline{F}\cdot\underline{ds} = ⊛$$

$$⊛ = \underbrace{\int_0^{x_B} t\sqrt{t^2}\,dt}_{\gamma_1} + \underbrace{\int_0^{y_B} t\cdot\sqrt{x_B^2+t^2}\,dt}_{\gamma_2} + \underbrace{\int_0^{z_B} t\cdot\sqrt{x_B^2+y_B^2+t^2}\,dt}_{\gamma} = \left[\frac{1}{3}(t^2)^{\frac{3}{2}}\right]_0^{x_B} + \left[\frac{1}{3}(x_B^2+t^2)^{\frac{3}{2}}\right]_0^{y_B} + \left[\frac{1}{3}(x_B^2+y_B^2+t^2)^{\frac{3}{2}}\right]_0^{z_B}$$

$$= \cancel{\frac{1}{3}x_B^3} + \cancel{\frac{1}{3}(x_B^2+y_B^2)^{3/2}} - \cancel{\frac{1}{3}x_B^3} + \frac{1}{3}(x_B^2+y_B^2+z_B^2) - \cancel{\frac{1}{3}(x_B^2+y_B^2)^{3/2}}$$

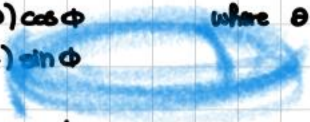
$$= \frac{1}{3}(x_B^2+y_B^2+z_B^2)$$

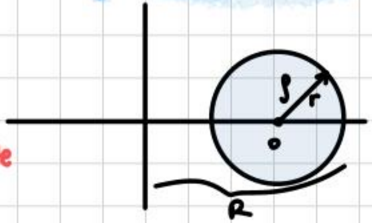
$$\Rightarrow g(B) = \frac{1}{3}(x^2+y^2+z^2)$$

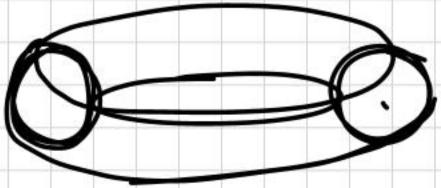
Exercise 7

$$\begin{cases} x(\theta,\phi) = (R+r\cos\theta)\cos\phi \\ y(\theta,\phi) = (R+r\cos\theta)\sin\phi \\ z(\theta,\phi) = r\sin\theta \end{cases}$$

where $\theta\in[0,2\pi]$, $\phi\in[0,2\pi]$







Introducing a Change of Variable

$$\begin{cases} x(\theta,\phi) = (R+\rho\cos\theta)\cos\phi \\ y(\theta,\phi) = (R+\rho\cos\theta)\sin\phi \\ z(\theta,\phi) = \rho\sin\theta \end{cases}$$

$\rho\in[0,r]$

$\theta\in[0,2\pi]$, $\varphi\in[0,2\pi]$

$\mathcal{U} = [0,2\pi]\times[0,2\pi]\times[0,r]$

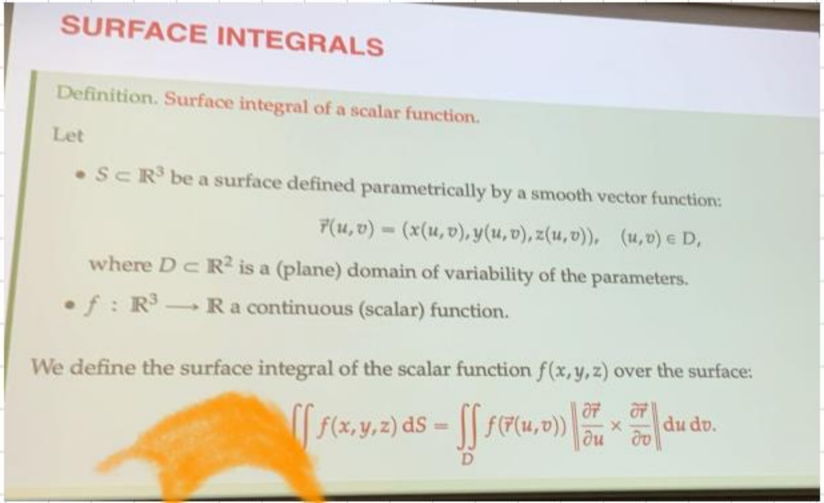
$\det J = \rho\left(R+\rho\cos\theta\right)$

$$\text{Volume}(V) = \iiint_V 1\,dV$$

$$= \iiint_{\mathcal{U}} \rho(R+\rho\cos\theta)\,d\rho\,d\theta\,d\varphi = \int_0^{2\pi} d\varphi \int_0^{2\pi}\left(\int_0^r \rho(R+\rho\cos\theta)\,d\rho\right)d\theta$$

$$= 2\pi\int_0^{2\pi}\left(\frac{r^2}{2}R + \frac{r^3}{3}\cos\theta\right)d\theta$$

$$= \cancel{2}\pi\ \frac{r^2R}{\cancel{2}}\,2\pi = r^2R\,2\pi^2$$



SURFACE INTEGRALS

Definition. Surface integral of a scalar function.

Let

- $S \subset \mathbb{R}^3$ be a surface defined parametrically by a smooth vector function:

$$\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v)), \quad (u,v) \in D,$$

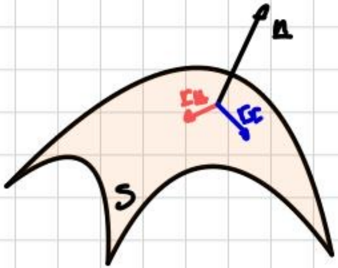
where $D \subset \mathbb{R}^2$ is a (plane) domain of variability of the parameters.

- $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ a continuous (scalar) function.

We define the surface integral of the scalar function $f(x,y,z)$ over the surface:

$$\iint f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v)) \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| du\,dv.$$

MEANING



$$r(\underline{u},\underline{v}) \begin{cases} \underline{r}_u = \dfrac{\partial r}{\partial u} \\ \underline{r}_v = \dfrac{\partial r}{\partial v} \end{cases}$$

Calculate the surface of the sphere with Radius R

$$\text{area}(S) = \iint_S 1\, dS$$

S ↳ not flat

$$\text{Area}(D) = \iint_D 1\, dD$$

in "flat"

@astrabocconi

@astrabocconi

@astrabocconi

ASTRA
BOCCONI